Introduction to "TGD as a Generalized Number Theory: Part II"

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1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K1].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

   Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A7]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes

for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of electromagnetic fields are nonvanishing. The correlations functions for weak fields are nonvanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement

theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question:super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

   TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

   TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A1] [B4, B2, B3]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral

is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B1]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4 \times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n \times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

1.2.1 TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A3, A6, A2, A5].

The identification of the space-time as a sub-manifold [A4, A9] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

1.2.2 TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

1.2.3 Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. ??** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle

(EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

1.3.1 Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

1.4.1 World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

[1] There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions

1.4.2 Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

1.4.3 WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H

or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

1.4.4 The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.5 Construction of scattering amplitudes

1.5.1 Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A8, A10, A11]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B+C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

1.5.2 Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

1.5.3 The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K10]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

1.6.1 The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

1.6.2 Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, M^8-H duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and M^8-H duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and M^8-H duality would map the algebraic physics in M^8 to the ordinary physics in $M^4\times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their M^8-H duals in M^8_c are defined for both reals and various p-adic number fields. This is true if they are

defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p = 3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

1.6.3 p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired "Universe as Computer" vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.

2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since $M^8 - H$ duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K9].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to "mind stuff", the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

1.6.4 Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

1.7.1 Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

1.7.2 Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M_c^4 point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M_c^4 \subset M_c^8$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{Re(E)^2 - Im(E)^2}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{Re(E)^2 - Im(E)^2}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} . \quad (1.1)$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

1.7.3 Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M_c^8$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M_c^4 \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique

and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

1.7.4 What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

1.7.5 Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M_c^8 and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M_c^8$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H_i^3, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X_i^3.

1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

1.8.1 Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$.

This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K18].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

1.8.2 Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K15, K16, K14]) support the view that dark matter might be a key player in living matter.

1.8.3 Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

1.9.1 Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [K23]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A7]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure coincides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   M^8-H duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [**?**].

1.9.2 Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their

boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [?]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it

seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t-m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

2 Bird's Eye of View about the Topics of "Quantum Physics as Number Theory: Part II"

The book "Quantum Physics as Number Theory: Part II" consists of 2 parts. The number theoretic vision of TGD has evolved slowly. I have decided to keep also the side tracks so that the chapters reflect this evolution of ideas.

1. In the first of the book M^8-H duality is discussed. Here M^8 refers to complexified octonions M_c^8. Physically M^8-H duality is a counterpart for the momentum position duality: indeed, M^8 can be interpreted the complexified co-tangent space of $M^4\times CP_2$ so that the points of M^8 are analogs of 8-momenta. In TGD the classical Noether momenta can be indeed complex: this is due the presence of Euclidian regions of space-time surface.

   This interpretation determines by a dimensional argument the form of the duality in M^4 degrees of freedom to as an analog of inversion (conformal symmetry) subject to the condition that the images of complex $M_c^4\subset M_c^8$ are real. In CP_2 degrees of freedom the assumption that the normal spaces are quaternionic and contain preferred 2-D complex subspace or integrable distribution of them implies that the normal spaces correspond to points of CP_2 and therefore can be mapped to points of CP_2. Clearly, the correspondence is non-local, which conforms with the Uncertainty Principle.

   M^8-H duality has a deeper mathematical meaning, which relates to Langlands correspondence between algebra and geometry. The duality states that the purely algebraic physics (no variational principle nor partial differential equations) based on algebraic surfaces in complexified M_c^8 is dual to the physics defined by preferred extremals (presumably minimal surfaces analogous to Bohr orbits and satisfying 4-D generalization of 2-D holomorphy) in $H=M^4\times CP_2$. Zero energy ontology, implied by holography in turn forced by 4-D general coordinate invariance, realizes quantum criticality, brings in an infinite number of constraints analogous to gauge conditions implying that space-time surfaces in H are analogs of Bohr orbits.

   The dynamics based on variational principle in H should be equivalent with purely algebraic physics in M^8 so that the 4-D surfaces in M_c^8 and H should be mappable to each other by

$M^8 - H$ duality. The fact that holomorphic solution ansatz works for any action principle which is general coordinate invariant and is constructible using induced geometry conforms with the number theoretical universality of number theoretic dynamics and its independence on variational principle.

Various structures and M^8 and H sides correspond to each other. In particular, the holography on H side requires a number theoretic counterpart of holography in M^8, naturally based on the associativity of the normal space (rather than tangent space, as it turns out) 4-D surfaces in M^8.

The holographic data must be determined by purely algebraic dynamics and are assumed to correspond for the simplest option to 3-surfaces at the mass shells of M_c^4 with mass squared value equal to a root of the polynomial P, which is a real polynomial with integer coefficients smaller than the degree of P. What makes this option unique is that the cognitive representations as points consisting of 4-momenta with components which are algebraic integers for the algebraic extensions associated with P can be said to explode.

The physical states correspond to many particle states, whose momenta corresponding to points of the mass shell are assumed to be Galois singlets and are therefore ordinary integers. This corresponds to particle in a box quantization. Galois confinement provides a universal mechanism for the formation of bound states.

2. In the second part of the book, I try to understand the relationship between $M^8 - H$ duality and Langlands correspondence. I think mathematically but I am not a theorem prover so that the considerations should be taken as a modest attempt of a physicist to see $M^8 - H$ duality as a physical principle having a concrete realization of Langlands duality.

3. The third part of the book involves number theoretical considerations. The role of Galois groups and finite fields in TGD is discussed. There is a chapter about McKay correspondence in the TGD framework and a proposal for the fusion of the number theoretic notions with those related to hyper-finite factors. Also a number theoretic view of emergence of chaos is proposed based on iteration of polynomials defining the holographic data for 4-surfaces in M^8.

3 Sources

The eight online books about TGD [K25, K24, K17, K12, K4, K11, K6, K20] and nine online books about TGD inspired theory of consciousness and quantum biology [K22, K3, K13, K2, K5, K7, K8, K19, K21] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://tinyurl.com/ybv8dt4n`) contains a lot of material about TGD. In particular, a TGD glossary at `http://tinyurl.com/yd6jf3o7`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://tinyurl.com/ycyrxj4o` founded by Lian Sidorov and in *Prespacetime Journal* (`http://tinyurl.com/ycvktjhn`), *Journal of Consciousness Research and Exploration* (`http://tinyurl.com/yba4f672`), and *DNA Decipher Journal* (`http://tinyurl.com/y9z52khg`), all of them founded by Huping Hu. One can find the list about the articles published at `http://tinyurl.com/ybv8dt4n`. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

3.1 PART I: $M^8 - H$ DUALITY

3.1.1 Does $M^8 - H$ duality relate the geometric view of TGD to octonionic algebraic geometry?: Part I

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

2. It will be shown how $M^8 - H$ duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in M^8 would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o_c decomposing as $o_c = q_c^1 + q_c^2 I^4$ and projected to a Minkowskian subspace M^8 of complexified O. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

   One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c\overline{q_c}$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and $M^8 - H$ correspondence could generalize.

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

   I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

3.1.2 Does $M^8 - H$ duality relate the geometric view of TGD to octonionic algebraic geometry?: Part II

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

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   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which

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   One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

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   I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

Also a sketchy proposal for the description of interactions is discussed.

1. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real

$P(t)$, whose octonionic extension defines the space-time variety X^4. The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices.

The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

2. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

   The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(\prod P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

3. The possibility of super octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

Scattering diagrams would be determined by points of space-time variety, which are in extension of rationals. In adelic physics the interpretation is as cognitive representations.

1. Cognitive representations are identified as sets of rational points for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [**?**]

2. Some aspects related to homology charge (Kähler magnetic charge) and genus-generation correspondence are discussed. Both topological quantum numbers are central in the proposed model of elementary particles and it is interesting to see whether the picture is internally consistent and how algebraic variety property affects the situation. Also possible problems related to $h_{eff}/h = n$ hierarchy []adelicphysics realized in terms of n-fold coverings of space-time surfaces are discussed from this perspective.

3.1.3 Does $M^8 - H$ duality relate the geometric view of TGD to octonionic algebraic geometry?: Part III

Cognitive representations are the basic topic of the third chapter related to $M^8 - H$ duality. Cognitive representations are identified as sets of points in extension of rationals for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces.

The notion is applied in various cases and the connection with $M^8 - H$ duality is rather loose.

1. Extensions of rationals are essentially coders of information. There the possible analogy of extensions of rationals with genes deserves discussion. Extensions, which are not extensions of extensions would be analogous to genes. The notion of conserved gene as number theoretical analogy for Galois extensions as the Galois group of extension which is normal subgroup of Galois extension.

2. The possible physical meaning of the notion of perfectoid introduced by Peter Scholze is discussed in the framework of p-adic physics. Extensions of p-adic numbers involving roots of the prime defining the extension are involved and are not considered previously in TGD framework. There there possible physical meaning deserves discussion.

3. The construction of cognitive representation reduces to a well-known mathematical problem of finding the points of space-time surface with embedding space coordinates in given extension of rationals. The work of Kim and Coates represents new ideas in this respect and there is a natural connection with TGD.

4. One expects that large cognitive representations are winners in the number theoretical fight for survival. Strong form of holography suggests that it is enough to consider cognitive representations restricted to string world sheets and partonic 2-surfaces. If the 2-surface possesses large group of symmetries acting in extension of rationals, one can have large cognitive representations as orbit of point in extension. Examples of highly symmetric 2-D surfaces are geodesic spheres assignable to partonic 2-surfaces and cosmic strings and elliptic curves assignable with string world sheets and cosmic strings.

5. Rationals and their extensions give rise to a unique discretizations of space-time surface (for instance) - cognitive representation - having interpretation in terms of finite measurement resolution. There are however many open questions. Should one allow only octonionic polynomials defined as algebraic continuations of real polynomials or should one allow also analytic functions and regard polynomials as approximations. Zeta functions are especially interesting analytic functions and Defekind zetas characterize extensions of rationals and one can pose physically motivated questions about them.

3.1.4 Breakthrough in understanding of $M^8 - H$ duality

A critical re-examination of $M^8 - H$ duality is discussed. $M^8 - H$ duality is one of the cornerstones of Topological Geometrodynamics (TGD). The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent or normal space and contains a complex or co-complex surface, it can be mapped to a 4-surface in $H = M^4 \times CP_2$.

Later emerged the idea that octonionic analyticity realized in terms of real polynomials P algebraically continued to polynomials of complexified octonion could fulfill the dream. The vanishing of the real part $Re_Q(P)$ (imaginary part $Im_Q(P)$) in the quaternionic sense would give rise to an associative (co-associative) space-time surface.

The realization of the general coordinate invariance motivated the notion of strong form of holography (SH) in H allowing realization of a weaker form of $M^8 - H$ duality by assuming that associativity/co-associativity conditions are needed only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits.

The outcome of the re-examination yielded both positive and negative surprises.

1. Although no interesting associative space-time surfaces are possible, every distribution of normal associative planes (co-associativity) is integrable.

2. Another positive surprise was that Minkowski signature is the only possible option. Equivalently, the image of M^4 as real co-associative subspace of O_c (complex valued octonion norm squared is real valued for them) by an element of local G_2 or rather, its subgroup $SU(3)$, gives a real co-associative space-time surface.

3. The conjecture based on naive dimensional counting, which was not correct, was that the polynomials P determine these 4-D surfaces as roots of $Re_Q(P)$. The normal spaces of these surfaces possess a fixed 2-D commuting sub-manifold or possibly their distribution allowing the mapping to H by $M^8 - H$ duality as a whole.

   If this conjecture were correct, strong form of holography (SH) would not be needed and would be replaced with extremely powerful number theoretic holography determining space-time surface from its roots and selection of real subspace of O_c characterizing the state of motion of a particle. erate

4. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [?, ?, ?] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [?, ?].

5. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X = 0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.

   The expectation was wrong! The equations $X = 0$ and $Y = 0$ involve the same(!) complex argument o^2_c as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X = 0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.

   There are solutions $X = 0$ and $Y = 0$ only if the two polynomials considered have a common a^2_c as a root! Also now the solutions are complex mass shells H^7_c.

How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .

1. The key observation is that G_2 acts as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space M^4 gives a co-associative four-surface as a real projection. Octonion analyticity would correspond to G_2 gauge transformation: this would realize the original idea about octonion analyticity.

2. A co-associative X^4_c satisfying also the conditions posed by the existence of $M^8 - H$ duality is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines $M^8 - H$ duality even if the co-associativity in geometric sense were not satisfied.

   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

   The group $SU(3)$ has interpretation as a Kac-Moody type analog of color group and the map defining space-time surface. This picture conforms with the H-picture in which gluon gauge potentials are identified as color gauge potentials. Note that at QFT limit the gauge potentials are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

3. Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \to M^4 \subset H$ from an identification to an almost inversion. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.

This picture combined with zero energy ontology leads also to a view about quantum TGD at the level of M^8. A local $SU(3)$ element defining 4-surface in M^8, which suggests a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

3.1.5 New findings related to the number theoretical view of TGD

The geometric vision of TGD is rather well-understood but there is still a lot of fog in the number theoretic vision.

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?

2. The detailed realization of $M^8 - H$ duality involves uncertainties: in particular, how the complexification of M^8 to M^8_c can be consistent with the reality of $M^4 \subset H$.

3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?

4. How unique is the choice of 3-D surfaces at the mass shells $H^3_m \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

These and many other questions motivated this article and led to the observation that the model geometries used in the classification of 3-manifolds seem to be rather closely related to the known space-time surfaces extremizing practically any general coordinate invariant action constructible in terms of the induced geometry.

The 4-surfaces in M^8 would define coupling constant evolutions for quantum states as analogs of and mappable to time evolutions at the level of H and obeying conservation laws associated with the dual conformal invariance analogous to that in twistor approach.

The momenta of fundamental fermions in the quantum state would be coded by the cusp singularities of 3-surfaces at the mass shells of M^8 and also its image in H provided by $M^8 - H$ duality. One can consider the possibility of $2 \rightarrow 3$ holography in which the boundaries of fundamental region of H^3/Γ is 2-D hyperbolic space H^2/Γ so that TGD could to high degree reduce to algebraic geometry.

3.1.6 A fresh look at $M^8 - H$ duality and Poincare invariance

$M^8 - H$ duality is a proposal to integrate geometric and number theoretic visions of TGD. $M^8 - H$ duality has several questionable features. For various reasons it seems that M^8 must be replaced with its complexification M^8_c interpreted as complexified octonions O_c. This however leads to several problems. The modified variant of $M^8 - H$ duality identifying M^8 as a quaternionic subspace of octonions O with a number theoretic norm defined by $Re(o^2)$, rather than $o\overline{o}$, solves these problems.

The proposal has been that octonionic polynomials $P(o)$ define the number theoretic holography. Their roots would define 3-D mass shells for which mass squared values are in general complex and the initial data for the holography would correspond to 3-surfaces at these mass shells. Also this assumption has problems. There is however no need for this assumption: the holography on the H side is induced by the $M^8 - H$ duality!

The hierarchy of polynomials defines a hierarchy of algebraic extensions defining an evolutionary hierarchy central for all applications of TGD and one must have it. Luckily, the recent realization that a generalized holomorphy realizes the holography at the H side as roots for pairs of holomorphic functions of complex (in generalized sense) coordinates of H comes to rescue. It can be strengthened by assuming that the functions form a hierarchy of pairs of polynomials.

Twistor lift strongly suggests that M^4 and space-time surfaces allow a Kähler structure and what I call Hamilton-Jacobi structure. These structures force a breaking of Poincare and even Lorentz invariance unless they are dynamically generated. It indeed turns out that $M^8 - H$ duality generates them dynamically.

3.1.7 Holography=holomorphy vision in relation to quantum criticality, hierarchy of Planck constants, and $M^8 - H$ duality

Holography = holomorphy vision generalizes the realization of quantum criticality in terms of conformal invariance. Holography = holomorphy vision provides a general explicit solution to the field equations determining space-time surfaces as minimal surfaces $X^4 \subset H = M^4 \times CP_2$. For the first option the space-time surfaces are roots of two generalized analytic functions P_1, P_2 defined in H . For the second option single analytic generalized analytic function defines X^4 as its root and as the base space of 6-D twistor twistor-surface X^6 in the twistor bundle $T(H) = T(M^4) \times TCP_2)$ identified as a zero section

By holography, the space-time surfaces correspond to not completely deterministic orbits of particles as 3-surfaces and are thus analogous to Bohr orbits. This implies zero energy ontology (ZEO) and to the view of quantum TGD as wave mechanics in the space of these Bohr orbits located inside a causal diamond (CD), which form a causal hierarchy. Also the construction of vertices for particle reactions has evolved dramatically during the last year and one can assign the vertices to partonic 2-surfaces.

$M^8 - H$ duality is a second key principle of TGD. $M^8 - H$ duality can be seen as a number theoretic analog for momentum-position duality and brings in mind Langlands duality. M^8 can be identified as octonions when the number-theoretic Minkowski norm is defined as $Re(o^2)$. The quaternionic normal space $N(y)$ of $y \in Y^4 \subset M^8$ having a 2-D commutative complex sub-space is mapped to a point of CP_2. Y^4 has Euclidean signature with respect to $Re(o^2)$. The points $y \in Y^4$ are lifted by a multiplication with a co-quaternionic unit to points of the quaternionic normal space $N(y)$ and mapped to $M^4 \subset H$ inversion.

This article discusses the relationship of the holography = holomorphy vision with the number theoretic vision predicting a hierarchy $h_{eff} = nh_0$ of effective Planck constants such that n corresponds to the dimension for an extension rationals (or extension F of rationals). How could this hierarchy follow from the recent view of $M^8 - H$ duality? Both realizations of holography = holomorphy vision assume that the polynomials involved have coefficients in an extension F of rationals Partonic 2-surfaces would represent a stronger form of quantum criticality than the generalized holomorphy: one could say islands of algebraic extensions F from the ocean of complex numbers are selected. For the P option, the fermionic lines would be roots of P and dP/dz inducing an extension of F in the twistor sphere. Adelic physics would emerge at quantum criticality and scattering amplitudes would become number-theoretically universal. In particular, the hierarchy of Planck constants and the identification of p-adic primes as ramified primes would emerge as a prediction.

Also a generalization of the theory of analytic functions to the 4-D situation is suggestive. The poles of cuts of analytic functions would correspond to the 2-D partonic surfaces as vertices at which holomorphy fails and 2-D string worlds sheets could correspond to the cuts. This provides a general view of the breaking of the generalized conformal symmetries and their super counterparts as a necessary condition for the non-triviality of the scattering amplitudes.

3.1.8 Does $M^8 - H$ duality reduce to local G_2 symmetry?

The idea of $\mathbb{M}^8 - \mathbb{H}$ duality has progressed through frustratingly many twists and turns and I have discussed several variants of $\mathbb{M}^8 - \mathbb{H}$ duality. There are 2 options, call them $\mathbb{T}$ and $\mathbb{N}$: either the local tangent space $\mathbb{T}$ or normal space $\mathbb{N}$ of $\mathbb{Y}^4 \subset \mathbb{M}^8$ is quaternionic and contains a complex subspace $\mathbb{C}$. This makes it possible to map $\mathbb{Y}^4 \subset \mathbb{M}^8$ to the space-time surface $\mathbb{X}^4 \subset \mathbb{H} = \mathbb{M}^4 \times \mathbb{CP}_2$. Which of them or possibly both? Any integrable distribution of quaternionic normal spaces $\mathbb{N}$ is allowed whereas for tangent spaces this is not the case. This led to a too hasty rejection of the $\mathbb{T}$ option.

The second problem relates to the lack of the concrete realization of the $\mathbb{M}^8 - \mathbb{H}$ duality. Is the $\mathbb{M}^8 - \mathbb{H}$ duality between 4-D surfaces in $\mathbb{M}^8$ and space-time surfaces in $\mathbb{H}$ or is it enough that only the 3-D holographic data in $\mathbb{H}$ are fixed by $\mathbb{M}^8-\mathbb{H}$ duality.

A modification of the original form of the $\mathbb{M}^8-\mathbb{H}$ duality formulated in terms of a real octonion analytic functions $f(o):\mathbb{O}-\mathbb{O}$ leads to a possible solution of these problems. All the conditions $f(o)=0$, $f(o)=1$ and $Imf)(o)=0$, and $Ref)(o)=0$ are invariant under local G_2 and the local G_2 acts as a dynamical spectrum generating symmetry group since $f\circ g_2=g_2\circ f$ holds true. The task reduces to that of finding 4-surfaces with constant quaternionic normal space $\mathbb{N}$ or tangent space $\mathbb{T}$. $\mathbb{Y}^4=\mathbb{E}^4\subset\mathbb{M}^8$ and $\mathbb{Y}^4=\mathbb{M}^4\subset\mathbb{M}^8$ provide the simplest examples of them. Local G_2 transformations give more general surfaces $\mathbb{Y}^4$.

The roots of $\mathrm{Im}(f)(o)=0$ *resp.* $\mathrm{Re}(f)(o)=0$ are unions $\cup_{o_0}\mathbb{S}^6(o_0)$ of 6-spheres, where o_0 is octonionic real coordinate o_0. The 3-surface $\mathbb{Y}^3=\mathbb{S}^6(o_0)\cap\mathbb{E}^4(o_0)=\mathbb{S}^3(o_0)$ defines holographic data for $\mathbb{Y}^4\subset\mathbb{E}^4(o_0)$ as its boundary. The union $\mathbb{Y}^3=\cup_{o_0}\mathbb{S}^6(o_0)\cap\mathbb{M}^4(o_0)=\cup_{o_0}\mathbb{S}^2(o_0)$ in turm defines holographic data for $\mathbb{Y}^4\subset\mathbb{M}^4(o_0)$ as its boundary. Therefore both the $\mathbb{N}$ - and $\mathbb{T}$ option can be realized.

One can choose the function $f(o)$ to be an analytic function of a hypercomplex coordinate of $\mathbb{M}^4$ and 3 complex coordinates of $\mathbb{M}^8$. The natural conjecture is that the image $\mathbb{X}^4$ of $\mathbb{Y}^4$ has the same property and satisfies holography = holomorphy principle.

The simultaneous roots of $\mathrm{Im}(f)(o)=0$ *resp.* $\mathrm{Re}(f)(o)=0$ are 6-spheres with fixed value of o_0 and the radius r_7 of $\mathbb{S}^6(o_0)$. Two 4-surfaces $\mathbb{Y}^4_1$ and $\mathbb{Y}^4_2$, both of type $\mathbb{N}$ or $\mathbb{T}$, and satisfying $\mathrm{Im}(f)(o)=0$ *resp.* $\mathrm{Re}(f)(o)=0$ along $\mathbb{S}^3(o_0)$ or $\mathbb{S}^2(o_0)$. This makes it possible to build Feynman diagram-like structures with lines which have Minkowskian or perhaps even Euclidean number theoretic metric signatures. At the vertices smoothness is violated and this supports the view that they give rise to exotic smooth structures as defects of the standard smooth structure.

3.1.9 About the construction of the scattering amplitudes using M^8-H duality

In TGD, point-like particles are replaced with 3-surfaces and these in turn with the analogs of Bohr orbits. M^8-H duality is the generalization of momentum-position duality and is now rather well understood. It however remains a mere academic mathematical construct unless it can be used to achieve some practical goal. The construction of scattering amplitudes is the basic dream of TGD and M^8-H duality gives hope of achieving this goal in terms of the TGD counterparts for the momentum space Feynman diagrams.

The notion of exotic smooth structure, having interpretation as an ordinary smooth structure with 3-D defects and possible only in 4-D space-time, is crucial. Fermions in H are free but fermion pair creation is possible at the defects at which fermion lines can turn backwards in time. Also a more general change of direction is possible. This makes the counterpart of fermionic Feynman diagrammatic extremely simple at the level of H. Only fermionic 2-vertices associated with 3-D geometric defects are needed. Fermionic interactions reduce to an 8-D Brownian motion in the induced classical fields and the singularities of the space-time surfaces at which minimal surface property fails define the location of the vertices.

The interactions of two space-time surfaces, identified in holography = holomorphy vision as 4-D generalized Bohr orbits, correspond geometrically to contact interactions at their intersections. If the Hamilton-Jacobi structures are the same, the intersections are 2-D strings world sheets. The edges of these string world sheets would contain the vertices.

In this article an attempt to formulate this picture at M^8 level by using a precise formulation of M^8-H duality is made.

3.2 PART II: LANGLANDS PROGRAM AND TGD

3.2.1 Langlands Program and TGD

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on one hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). If TGD can be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful. I of course hasten to confess that I am not number theorists nor group theorists and that the following considerations are just speculations inspired by TGD.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also a generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW in terms of hyper-finite factors of type II_1 in turn inspires further generalization of the notion of embedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of embedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

1. Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.

2. The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..

3. TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

4. Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more

precisely to the regions of partonic 2-surface associated with braids. $Sp(2g,R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

1. Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

2. There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

   (a) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

   (b) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

   (c) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \rightarrow H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s - s_n)/(1 - s + s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

3.2.2 Langlands Program and TGD: Years Later

Langlands correspondence is for mathematics what unified theories are for physics. The number theoretic vision about TGD has intriguing resemblances with number theoretic Langlands program. There is also geometric variant of Langlands program. I am of course amateur and do not have grasp about the mathematical technicalities and can only try to understand the general ideas and related them to those behind TGD. Physics as geometry of WCW ("world of classical worlds") and physics as generalized number theory are the two visions about quantum TGD: this division brings in mind geometric and number theoretic Langlands programs. This motivates re-consideration of Langlands program from TGD point of view. I have written years ago a chapter about this earlier but TGD has evolved considerably since then so that it is time for a second attempt to understand what Langlands is about.

By Langlands correspondence the representations of $G \rtimes Gal$ and G should correspond to each other. The analogy with the representations of Lorentz group suggests that the representations of G should have "spin" for some compact subgroup acting from left or right such that the dimension of this representation is same as the representation of non-commutative Galois group.

Automorphic functions are indeed typically functions in G, which reduce to a function invariant under left and/or right action of a compact or even discrete subgroups H_1 and H_2 or more generally, belong to a finite-dimensional unitary representation of $H_1 \times H_2$ in $H_1\backslash G/H_2$. Therefore they can be said to have $H_1 \times H_2$ quantum numbers analogous to spin if interpreted as "field modes" in the space of double cosets $H_1 g H_2$. This would conform with the vision about physics as generalized number theory. If I have understood correctly, the question is whether a finite-dimensional representation of H_1 or H_2 could correspond to a finite-dimensional representation of Galois group at the number theory side.

3.2.3 Some New Ideas Related to Langlands Program *viz.* TGD

Langlands' program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Langlands program is described by Edward Frenkel as a kind of grand unified theory of mathematics.

In the TGD framework, $M^8 - M^4 \times CP_2$ duality assigns to a rational polynomial a set of mass shells H^3 in $M^4 \subset M^8$ and by associativity condition a 4-D surface in M^8, and its it to $H = M^4 \times CP_2$. $M^8 - M^4 \times CP_2$ means that number theoretic vision and geometric vision of physics are dual or at least complementary. This vision could extend to a trinity of number theoretic, geometric and topological views since geometric invariants defined by the space-time surfaces as Bohr orbit-like preferred extremals could serve as topological invariants.

Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC), which is formulated as a correspondence between the quantum states in the "world of classical worlds" (WCW) characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces.

L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via Galois group representation partition function. QCC would define a kind of closed loop giving rise to a hierarchy.

If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.

One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework. Elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters. Multiple residue integral is proposed as a way to obtain L-functions defining space-time surfaces.

One challenge is to construct Riemann zeta and the associated ξ function and the Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

This construction leads to the question whether also finite fields could play a fundamental role in the number theoretic vision. Prime polynomial with prime order $n = p$ and integer coefficients smaller than $n = p$ can be regarded as a polynomial in a finite field. If it is irreducible, it defines an infinite prime. The proposal is that all physically allowed polynomials are constructible as functional composites of these.

3.2.4 About Langlands correspondence in the TGD framework

The considerations of this article were inspired by an interview of Edward Frenkel relating to the Langlands correspondence and led to a considerably more detailed understanding of how number theoretic and geometric Langlands correspondences emerge in the TGD framework from number theoretic universality, holography=holomorphy vision leading to a general solution of field equations based on the generalization of holomorphy, and $M^8 - H$ duality relating geometric and number theoretic visions of TGD.

The space-time surfaces are realized as roots for a pair (P_1, P_2) of holomorphic polynomials of four generalized complex coordinates of $H = M^4 \times CP_2$. In this view space-time surfaces are representations of the function field of generalized polynomial pairs in H and can be regarded as numbers with product induced from the product $(P_1, Q) * (P_2, Q) = (P_1 P_2, Q)$. Same applies to other arithmetic operations.

A proposal for how to count the number of roots of the $(P_1, P_2) = (0, 0)$, when the arguments are restricted to a finite field in terms of modular forms defined at the hyperboloid $H^3 \times CP_2 \subset M^4 \times CP_2$. The geometric variant of the Galois group as a group mapping different roots for a polynomial pair (P_1, P_2) identifiable as regions of the space-time surface (minimal surface) would be in terms of homomorphisms of H.

The interpretation of space-time surfaces as numbers leads to a general construction recipe for quantum states in terms of a geometric analog $(P_1, Q) * (P_2, Q) = (P_1 P_2, Q)$ of a tensor product with the property that the product corresponds to a union of the surfaces $(P_1, Q) = (0, 0)$ and $(P_2, Q) = 0$. The asymmetry between P_i and Q has a physical interpretation. The interactions would be associated with stable intersection points of these surfaces and a deformation making $P_1 P_2$ irreducible replacing them with wormhole contacts is suggestive. One could say that TGD is exactly solvable at both classical space-time level and quantum level. The geometric Langlands correspondence extends to a trinity between number theory, geometry and physics.

3.2.5 A more detailed view about the TGD counterpart of Langlands correspondence

In TGD, geometric and number theoretic visions of physics are complementary. This complementarity is analogous to momentum position duality of quantum theory and implied by the replacement of a point-like particle with 3-surface, whose Bohr orbit defines space-time surface.

At a very abstract level this view is analogous to Langlands correspondence. The recent view of TGD involving an exact algebraic solution of field equations based on holography= holomorphy vision allows to formulate the analog Langlands correspondence in 4-D context rather precisely. This requires a generalization of the notion of Galois group from 2-D situation to 4-D situation: there are 2 generalizations and both are required.

1. The first generalization realizes Galois group elements, not as automorphisms of a number field, but as analytic flows in $H = M^4 \times CP_2$ permuting different regions of the space-time surface identified as roots for a pair $f = (f_1, f_2)$ of pairs $f = (f_1, f_2) : H \rightarrow C^2$, $i = 1, 2$. The functions f_i are analytic functions of one hypercomplex and 3 complex coordinates of H.

2. Second realization is for the spectrum generating algebra defined by the functional compositions $g \circ f$, where $g : C^2 \rightarrow C^2$ is analytic function of 2 complex variables. The interpretation is as a cognitive hierarchy of function of functions of .... and the pairs (f_1, f_2) which do not allow a composition of form $f = g \circ h$ correspond to elementary function and to the lowest levels of this hierarchy, kind of elementary particles of cognition. Also the pairs g can be expressed as composites of elementary functions.

   If g_1 and g_2 are polynomials with coefficients in field E identified as an extension of rationals, one can assign to $g \circ f$ root a set of pairs (r_1, r_2) as roots $f_1, f_2) = (r_1, r_2)$ and r_i are algebraic numbers defining disjoint space-time surfaces. One can assign to the set of root pairs the analog of the Galois group as automorphisms of the algebraic extension of the field E appearing as the coefficient field of (f_1, f_2) and (g_1, g_2). This hierarchy leads to the idea that physics could be seen as an analog of a formal system appearing in Gödel's theorems and that the hierarchy of functional composites could correspond to a hierarchy of meta levels in mathematical cognition.

3. The quantum generalization of integers, rationals and algebraic numbers to their functional counterparts is possible for maps $g : C^2 \rightarrow C^2$. The counterpart of the ordinary product is functional composition $\circ$ for maps g. Degree is multiplicative in $\circ$. In sum, call it $+_e$, the degree should be additive, which leads to the identification of the sum $+_e$ as an element-wise product. The neutral element $1_\circ$ of $\circ$ is $1_\circ = Id$ and the neutral element 0_e of $+_e$ is the ordinary unit $0_e = 1$.

   The inverse corresponds to g^{-1} for $\circ$, which in general is a many-valued algebraic function and to $1/g$ for $times$. The maps g, which do not allow decomposition $g = h \circ i$, can be identified as functional primes and have prime degree. $f : H \rightarrow C^2$ is prime if it does not allow composition $f = g \circ h$. Functional integers are products of functional primes g_p.

   The non-commutativity of $\circ$ could be seen as a problem. The fact that the maps g act like operators suggest that the functional primes g_p in the product commute. Functional integers/rationals can be mapped to ordinary by a morphism mapping their degree to integer/rational.

4. One can define functional polynomials $P(X)$, quantum polynomials, using these operations. In $P(X)$, the terms $p_n \circ X^n$, p_n and X should commute. The sum $\sum_e p_n X^n$ corresponds to $+_e$. The zeros of functional polynomials satisfy the condition $P(X) = 0_e = 1$ and give as solutions roots X_k as functional algebraic numbers. The fundamental theorem of algebra generalizes at least formally if X_k and X commute. The roots have representation as a space-time surface. One can also define functional discriminant D as the $\circ$ product of root differences $X_k -_e X_l$, with $-_e$ identified as element-wise division and the functional primes dividing it have space-time surface as a representation.

What about functional p-adics?

1. The functional powers $g_p^{\circ k}$ of primes g_p define analogs of powers of p-adic primes and one can define a functional generalization of p-adic numbers as quantum p-adics. The coefficients X_k $X_k \circ g_p^k$ are polynomials with degree smaller than p. The sum $+_e$ so that the roots are disjoint unions of the roots of $X_k \circ g_p^k$.

2. Large powers of prime appearing in p-adic numbers must approach 0_e with respect to the p-adic norm so that g_P^n must effectively approach Id with respect to $\circ$. Intuitively, a large n in g_P^n corresponds to a long p-adic length scale. For large n, g_P^n cannot be realized as a

space-time surface in a fixed CD. This would prevent their representation and they would correspond to 0_e and Id. During the sequence of SSFRs the size of CD increases and for some critical SSFRs a new power can emerge to the quantum p-adic.

3. Universal Witt polynomials W_n define an alternative representation of p-adic numbers reducing the multiplication of p-adic numbers to elementwise product for the coefficients of the Witt polynomial. The roots for the coefficients of W_n define space-time surfaces: they should be the same as those defined by the coefficients of functional p-adics.

There are many open questions.

1. The question whether the hierarchy of infinite primes has relevance to TGD has remained open. It turns out that the 4 lowest levels of the hierarchy can be assigned to the rational functions $f_i : H \to C^2$, $i = 1, 2$ and the generalization of the hierarchy can be assigned to the composition hierarchy of prime maps g_p.

2. Could the transitions $f \to g \circ f$ correspond to the classical non-determinism in which one root of g is selected? If so, the p-adic non-determinism would correspond to classical non-determinism. Quantum superposition of the roots would make it possible to realize the quantum notion of concept.

3. What is the interpretation of the maps g^{-1} which in general are many-valued algebraic functions if g is rational function? g increases the complexity but g^{-1} preserves or even reduces it so that its action is entropic. Could selection between g and g_{-1} relate to a conscious choice between good and evil?

4. Could one understand the p-adic length scale hypothesis in terms of functional primes. The counter for functional Mersenne prime would be g_2^n/g_1, where division is with respect to elementwise product defining $+_e$? For g_2 and g_3 and also their iterates the roots allow analytic expression. Could primes near powers of g_2 and g_3 be cognitively very special?

3.2.6 Space-time surfaces as numbers, Turing and Gödel, and mathematical consciousness

In this chapter a rather detailed view about the realization of Langlands correspondence (LC) is discussed. The geometric and function field versions naturally correspond to each other and the LC itself boils down to the condition that cobordisms for the function pairs (f_1, f_2) defining the space-time surfaces as union of regions defined by their roots are realized as flows in the infinite-D symmetry group permuting space-time regions as roots of a function pair (f_1, f_2) acting in the "world of classical worlds" (WCW) consisting of space of space-time surfaces satisfying holography = holomorphy principle.

Space-time surfaces form an algebra with respect to multiplication and this algebra decomposes to a union of number fields. This suggest a dramatic revision of what computation means physically. The standard view of computation as a construction of arithmetic functions is replaced with a physical picture in which space-times as 4-surfaces have interpretation as almost deterministic computations. Space-time surfaces allow arithmetic operations and also the counterparts of functional composition and iteration are well-defined. This would suggest a dramatic generalization of the computational paradigm and it is interesting to ponder what this might mean.

This also leads to a vision about the geometric correlates of arithmetic and even more general mathematical consciousness based on the vision about space-time surfaces as generalized numbers and providing also a representation of the ordinary complex numbers.

3.2.7 Gödel, Lawvere, and TGD

The tweets of Curt Jaimungal inspired an attempt to understand Gödel's incompleteness theorem and related constructions from the TGD point of view. The basic idea is that the laws of physics, as they are formulated in the TGD framework, can be regarded as analogs for the axioms of a formal system.

1. Space-time surface which by holography= holomorphy vision is analogous to a Bohr orbit of particles represented as a 3-surface is analogous to a theorem. The slight classiocal non-determinism implies that there are several Bohr orbits associated with the same 3-surface at the passive boundary of causal diamond remaining un-affected in the sequence of small state function reductions (TGD counterpart of the Zeno effect).

   The holographic data would be in the role of the assumptions of a theorem, which need not to be proved and reduced to axioms, and the Bohr orbits would correspond to theorems deducible from these assumptions.

2. The adelization of physics means that real space-time surfaces obtained using extension of E of rationals are extended to adelic space-time surfaces. The p-adic analogs of the space-time surface would be correlates for cognition and cognitive representations correspond to the intersections of the real space-time surface and its p-adic variants with points having Hamilton-Jacobi coordinates in E.

3. Concerning Gödel, the most important question is how self references as a metamathematical notion is realized: how space-time surfaces can represent analogs of statements about space-time surfaces. In this framework, meta level could correspond to the maps $g : C^2 \rightarrow C^2$ mapping the function pairs $f = (f_1, f_2) : H \rightarrow C^2$ defining space-time surfaces as their roots to the composites $g \circ f$. g should act trivially at the passive boundary (PB) of CD. One can construct hierarchies of these composites.

4. Second realization would be in terms of the hierarchy of infinite primes analogous to a repeated second quantization of a supersymmetric arithmetic quantum field theory for an extension E of rationals. Also the Fock basis of WCW spinor fields relates to WCW like the set of statements about statements to the set of statements.

The TGD counterpart for the action of an object of a formal system acting as a morphism on another object is the action of the space-time surface X^4 to another surface Y^4 and vice versa. This correspond to the interaction of the Bohr orbits of X^3 and Y^3 involving a temporary formation of a connected 3-surface as an intermediate state (monopole flux tubes could connect the 3- surfaces).This action is highly unique and fixed apart from the weak classical non-determinism. The interaction would be analogous to sensory perception. The time reversal of this sensory perception involving two BSFRs changing the arrow of time would correspond to motor action. In this view, the infinite self reflection hierarchy is replaced with a finite SSFRs and self is a dynamical object.

3.2.8 Holography= holomorphy vision: analogues of elliptic curves and partonic orbits

Holography= holomorphy principle allows to solve the extremely nonlinear partial differential equations for the space-time surfaces exactly by reducing them to algebraic equations involving an identically vanishing contraction of two holomorphic tensors of different types. In this article, space-time counterparts for elliptic curves and doubly periodic elliptic functions, in particular Weierstrass function, are considered as an application of the method.

Calabi-Yau manifolds are n-complex-dimensional generalizations of elliptic surfaces and recently found to emerge in a quantum field theory based model for energy production in the scattering of blackholes. This motivates the question whether the holography= holomorphy principle could allow the appearance of hyperbolic variant of K3 surface as 2-complex dimensional CY manifolds as analogs of lattice cells for periodic space-time surfaces, perhaps as nonlinear generalizations of 4-dimensional plane waves with the periods serving as counterparts of wave vectors. The space-time surfaces are identified as intersections of 2 6-D surfaces: could they correspond in some cases to hypercomplex variants of 3-complex-dimensional Calabi-Yau manifolds?

Partonic orbits as surfaces at which the Minkowskian signature of the metric transforms from Minkowskian to Euclidian are identified as orbits of partonic 2-surfaces. Holography= holomorphy vision allows to transform $det(g_4) = 0$ condition to a set 1-D Virasoro conditions labelled by points of the partonic 2-surface.

3.2.9 What could 2-D minimal surfaces teach about TGD?

In the TGD Universe space-time surfaces within causal diamonds (CDs) are fundamental objects.

1. M^8-H duality means that one can interpret the space-time surfaces in two manners: either as an algebraic surface in complexified M^8 or as minimal surfaces in $H = M^4 \times CP_2$. M^8-H duality maps these surfaces to each other.

2. Minimal surface property holds true outside the frame spanning minimal surface as 4-D soap film and since also extremal of Kähler action is in question, the surface is analog of complex surface. The frame is fixed at the boundaries of the CD and dynamically generated in its interior. At frame the isometry currents of volume term and Kähler action have infinite divergences which however cancel so that conservation laws coded by field equations are true. The frames serve as seats of non-determinism.

3. At the level of M^8 the frames correspond to singularities of the space-time surface. The quaternionic normal space is not unique at the points of a d-dimensional singularity and their union defines a surface of CP_2 of dimension $d_c = 4 - D < d$ defining in H a blow up of dimension d_c.

In this article, the inspiration provided by 2-D minimal surfaces is used to deepen the TGD view about space-time as a minimal surface and also about $M^8 - H$ duality and TGD itself.

1. The properties of 2-D minimal surfaces encourage the inclusion of the phase with a vanishing cosmological constant Λ phase. This forces the extension of the category of real polynomials determining the space-time surface at the level of M^8 to that of real analytic functions. The interpretation in the framework of consciousness theory would be as a kind of mathematical enlightenment, transcendence also in the mathematical sense.

2. $\Lambda > 0$ phases associated with real polynomials as approximations of real analytic functions would correspond to a hierarchy of inclusions of hyperfinite-factors of type II_1 realized as physical systems and giving rise to finite cognition based on finite-D extensions of rationals and corresponding extensions of p-adic number fields.

3. The construction of 2-D periodic minimal surfaces inspires a construction of minimal surfaces with a temporal periodicity. For $\Lambda > 0$ this happens by gluing copies of minimal surface and its mirror image together and for $\Lambda = 0$ by using a periodic frame.

   A more general engineering construction using different basic pieces fitting together like legos gives rise to a model of logical thinking with thoughts as legos. This also allows an improved understanding of how M^8-H duality manages to be consistent with the Uncertainty Principle (UP).

4. At the physical level, one gains a deeper understanding of the space-time correlates of particle massivation and of the TGD counterparts of twistor diagrams. Twistor lift predicts M^4 Kähler action and its Chern-Simons implying CP breaking. This part is necessary in order to have particles with non-vanishing momentum in the $\Lambda = 0$ phase.

3.3 PART III: FURTHER NUMBER THEORETICAL CONSIDERATIONS

3.3.1 About the role of Galois groups in TGD framework

This article was inspired by the inverse problem of Galois theory. Galois groups are realized as number theoretic symmetry groups realized physically in TGD a symmetries of space-time surfaces. Galois confinement as an analog of color confinement is proposed in TGD inspired quantum biology.

Two instances of the inverse Galois problem, which are especially interesting in TGD, are following:

Q1: Can a given finite group appear as Galois group over Q? The answer is not known.

Q2: Can a given finite group G appear as a Galois group over some EQ? Answer to Q2 is positive as will be found and the extensions for a given G can be explicitly constructed.

The TGD based formulation based on $M^8 - H$ duality in which space-time surface in complexified M^8 are coded by polynomials with rational coefficients involves the following open question.

Q: Can one allow only polynomials with coefficients in Q or should one allow also coefficients in EQs?

The idea allowing to answer this question is the requirement that TGD adelic physics is able to represent all finite groups as Galois groups of Q or some EQ acting physical symmetry group.

If the answer to **Q1** is positive, it is enough to have polynomials with coefficients in Q. It not, then also EQs are needed as coefficient fields for polynomials to get all Galois groups. The first option would be the more elegant one.

In the sequel the inverse problem is considered from the perspective of TGD. Galois groups, in particular simple Galois groups, play a fundamental role in the TGD view of cognition. The TGD based model of the genetic code involves in an essential manner the groups A_5 (icosahedron), which is the smallest simple and non-commutative group, and A_4 (tetrahedron). The identification of these groups as Galois groups leads to a more precise view about genetic code.

3.3.2 Finite Fields and TGD

TGD involves geometric and number theoretic physics as complementary views of physics. Almost all basic number fields: rationals and their algebraic extensions, p-adic number fields and their extensions, reals, complex number fields, quaternions, and octonions play a fundamental role in the number theoretical vision of TGD.

Even a hierarchy of infinite primes and corresponding number fields appears. At the first level of the hierarchy of infinite primes, the integer coefficients of a polynomial Q defining infinite prime have no common prime factors. $P = Q$ hypothesis states that the polynomial P defining space-time surface is identical with a polynomial Q defining infinite prime at the first level of hierarchy.

However, finite fields, which appear naturally as approximations of p-dic number fields, have not yet gained the expected preferred status as atoms of the number theoretic Universe. Also additional constraints on polynomials P are suggested by physical intuition.

Here the notions of prime polynomial and concept of infinite prime come to rescue. Prime polynomial P with prime order $n = p$ and integer coefficients smaller than p can be regarded as a polynomial in a finite field. The proposal is that all physically allowed polynomials are constructible as functional composites of prime polynomials satisfying $P = Q$ condition.

One of the long standing mysteries of TGD is why preferred p-adic primes, characterizing elementary particles and even more general systems, satisfy the p-adic length scale hypothesis. The proposal is that p-adic primes correspond to ramified primes as factors of discriminant D of polynomial $P(x)$. $D = P$ condition reducing discriminant to a single prime is an attractive hypothesis for preferred ramified primes. $M^8 - H$ duality suggests that the exponent $exp(K)$ of Kähler function corresponds to a negative power D^{-k}. Spin glass character of WCW suggests that the preferred ramified primes for, say prime polynomials of a given degree, and satisfying $D = P$, have an especially large degeneracy for certain ramified primes P, which are therefore of a special physical importance.

3.3.3 McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

This article deals with two questions.

1. The ideas related to topological quantum computation suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of state space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum.

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Or could one replace the roots labelling states with representations? Or could even the coefficient field for state space be replaced with the representations? Could one speak about quantum variants of state spaces?

Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums in quantum-classical correspondence, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ to those of $SL(n, C)$ at least. Is there a deep connection between finite subgroups of $SL(n, C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

In the TGD framework $M^8 - H$ duality relates number theoretic and differential geometric views about physics: could it provide some understanding of this mystery? The proposal is that for cognitive representations associated with extended Dynkin diagrams (EEDs), Galois group Gal acts as Weyl group on McKay diagrams defined by irreps of the isotropy group Gal_I of given root of a polynomial which is monic polynomial but with roots replaced with direct sums of irreps of Gal_I. This could work for p-adic number fields and finite fields. One also ends up with a more detailed view about the connection between the hierarchies of inclusion of Galois groups associated with functional composites of polynomials and hierarchies of inclusions of hyperfinite factors of type II_1 assignable to the representation of super-symplectic algebra.

3.3.4 Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole

The theoretical framework behind TGD involves several different strands and the goal is to unify them to a single coherent whole. TGD involves number theoretic and geometric visions about physics and $M^8 - H$ duality, analogous to Langlands duality, is proposed to unify them. Also quantum classical correspondence (QCC) is a central aspect of TGD. One should understand both the $M^8 - H$ duality and QCC at the level of detail.

The following mathematical notions are expected to be of relevance for this goal.

1. Von Neumann algebras, call them M, in particular hyperfinite factors of type II_1 (HFFs), are in a central role. Both the geometric and number theoretic side, QCC could mathematically correspond to the relationship between M and its commutant M'.

   For instance, symplectic transformations leave induced Kähler form invariant and various fluxes of Kähler form are symplectic invariants and correspond to classical physics commuting with quantum physics coded by the super symplectic algebra (SSA). On the number theoretic side, the Galois invariants assignable to the polynomials determining space-time surfaces are analogous classical invariants.

2. The generalization of ordinary arithmetics to quantum arithmetics obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ allows us to replace the notions of finite and p-adic number fields with their quantum variants. The same applies to various algebras.

3. Number theoretic vision leads to adelic physics involving a fusion of various p-adic physics and real physics and to hierarchies of extensions of rationals involving hierarchies of Galois groups involving inclusions of normal subgroups. The notion of adele can be generalized by replacing various p-adic number fields with the p-adic representations of various algebras.

4. The physical interpretation of the notion of infinite prime has remained elusive although a formal interpretation in terms of a repeated quantization of a supersymmetric arithmetic QFT is highly suggestive. One can also generalize infinite primes to their quantum variants. The proposal is that the hierarchy of infinite primes generalizes the notion of adele.

The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves f 3 kinds of algebras A; supersymplectic isometries SSA acting on $\delta M^4_+ \times CP_2$, affine algebras Aff acting on light-like partonic orbits, and isometries I of light-cone boundary δM^4_+, allowing hierarchies A_n.

The braided Galois group algebras at the number theory side and algebras $\{A_n\}$ at the geometric side define excellent candidates for inclusion hierarchies of HFFs. M^8-H duality suggests that n corresponds to the degree $n$of the polynomial P defining space-time surface and that the n roots of P correspond to n braid strands at H side. Braided Galois group would act in A_n and hierarchies of Galois groups would induce hierarchies of inclusions of HFFs. The ramified primes of P would correspond to physically preferred p-adic primes in the adelic structure formed by p-adic variants of A_n with $+$ and $\times$ replaced with $\oplus$ and $\otimes$.

3.3.5 Could quantum randomness have something to do with classical chaos?

Tim Palmer has proposed that classical chaos and quantum randomness might be related. It came as a surprise to me that these to notions could a have deep relationship in TGD framework.

1. Strong form of Palmer's idea stating that quantum randomness reduces to classical chaos certainly fails but one can consider weaker forms of the idea. Even these variants fail in Copenhagen interpretation since strictly speaking there is no classical reality, only wave function coding for the knowledge about the system. Bohr orbits should be more than approximation and in TGD framework space-time surface as preferred extremal of action is analogous to Bohr orbit and classical physics defined by Bohr orbits is an exact part of quantum theory.

2. In the zero energy ontology (ZEO) of TGD the idea works in weaker form and has very strong implications for the more detailed understanding of ZEO and $M^8-M^4\times CP_2$ duality. Ordinary ("big") state functions (BSFRs) meaning the death of the system in a universal sense and re-incarnation with opposite arrow of time would involve quantum criticality accompanied by classical chaos assignable to the correspondence between geometric time and subjective time identified as sequence of "small" state function reductions (SSFRs) as analogs of weak measurements. The findings of Minev et al give strong support for this view and Libet's findings about active aspects of consciousness can be understood if the act of free will corresponds to BSFR.

M^8 picture identifies 4-D space-time surfaces X^4 as roots for "imaginary" or "real" part of octonionic polynomial P_2P_1 obtained as a continuation of real polynomial $P_2(L-r)P_1(r)$, whose arguments have origin at the the tips of B and A and roots a the light-cone boundaries associated with tips. Causal diamond (CD) is identified intersection of future and past directed light-cones light-cones A and B. In the sequences of SSFRs $P_2(L-r)$ assigned to B varies and $P_1(r)$ assigned to A is unaffected. L defines the size of CD as distance $\tau=2L$ between its tips.

Besides 4-D space-time surfaces there are also brane-like 6-surfaces corresponding to roots $r_{i,k}$ of $P_i(r)$ and defining "special moments in the life of self" having $t_i=r_{i,k}$ ball as M^4_+ projection. The number of roots and their density increases rapidly in the sequence of SSFRs. The condition that the largest root belongs to CD gives a lower bound to it size L as largest root. Note that L increases.

Concerning the approach to chaos, one can consider three options.

Option I: The sequence of steps consisting of unitary evolutions followed by SSFR corresponds to a functional factorization at the level of polynomials as sequence $P_2=Q_1\circ Q_2\circ ...Q_n$. If the size of CD is assumed to increase, also the tip of active boundary of CD must shift so that the argument of P_2 $r-L$ is replaced in each iteration step to with updated argument with larger value of L.

Option II: A completely unexpected connection with the iteration of analytic functions and Julia sets, which are fractals assigned also with chaos interpreted as complexity emerges. In a reasonable approximation quantum time evolution by SSFRs could be induced by an iteration of a polynomial or even an analytic function: $P_2=P_2\rightarrow P_2^{\circ 2}\rightarrow ....$ For $P_2(0)=0$ the roots of the iterate consists of inverse images of roots of P_2 by $P_2^{\circ -k}$ for $k=0,...,N-1$.

Suppose that M^8 and X^4 are complexified and thus also $t=r$ and "real" X^4 is the projection of X^4_c to real M^8. Complexify also the coefficients of polynomials P. If so, the Mandelbrot and Julia sets (`http://tinyurl.com/cplj9pe` and `http://tinyurl.com/cvmr83g`) characterizing fractals would have a physical interpretation in ZEO.

One approaches chaos in the sense that the $N-1$:th inverse images of the roots of P_2 belonging to filled Julia set approach to points of Julia set of P_2 as the number N of iterations increases. Minimal L would increase with N if CD is assumed to contain all roots. The density of the roots in Julia set increases near L since the size of CD is bounded by the size Julia set. One could perhaps say that near the $t = L$ in the middle of CD the life of self when the size of CD has become almost stationary, is the most intensive.

Option III: A conservative option is to consider also real polynomials $P_2(r)$ with real argument r. Only non-negative real roots r_n are of interest whereas in the general case one considers all values of r. For a large N the new roots with possibly one exception would approach to the real Julia set obtained as a real projection of Julia set for complex iteration.

How the size L of CD is determined and when can BSFR occur?

Option I: If L is minimal and thus given by the largest (non-exceptional) root of iterate of P_2 in Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). L should smaller than the sizes of Julia sets of both$A$ and B since the iteration gives no roots outside Julia sets.

Could BSFR become probable when L as the largest allowed root for iterate P_2 is larger than the size of Julia set of A? There would be no more new "special moments in the life of self" and this would make death (in universal sense) and re-incarnation with opposite arrow of time probable. The size of CD could decrease dramatically in the first iteration for P_1 if it is determined as the largest allowed root of P_1: the re-incarnated self would have childhood.

Option II: The size of CD could be determined in SSFR statistically as an allowed root of P_2. Since the density of roots increases, one would have a lot of choices and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time. BSFR would occur for the same reason as for the first option.

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