Infinite Primes and Consciousness

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February 2, 2024

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Contents

Abstract

Infinite primes are besides p-adicization and the representation of space-time surface as an associative (co-associative) sub-manifold of hyper-octonionic space, basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible. Infinite primes generate wild philosophical speculations involved and the fate of speculations is usually sad. There are also amazing analogies with basic quantum physics, which make me to take infinite primes seriously.

1. Why infinite primes are unavoidable?

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. The first view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D hyper-octonions make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic embedding space.
2. The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

4. Infinite primes as a bridge between quantum and classical?

An important stimulus came from the observation stimulated by algebraic number theory. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes whereas hitherto it was possible to construct explicitly only what might be called generating infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers).

Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

5. Various equivalent characterizations of space-times as surfaces

One can imagine several number-theoretic characterizations of the space-time surface.

1. The approach based on octonions and quaternions suggests that space-time surfaces might correspond to associative or hyper-quaternionic surfaces of hyper-octonionic embedding space.
2. Space-time surfaces could be seen as absolute minima of the Kähler action. The challenge is to prove that this characterization is equivalent with the number theoretical dynamics,

6. The representation of infinite complex-octonionic primes as 4-surfaces

The difficulties caused by the Euclidian metric signature of the number theoretical norm forced to give up the idea that space-time surfaces could be regarded as quaternionic submanifolds of octonionic space, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart for the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The notions of hyper-quaternionic and octonionic manifold make sense but it is implausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces can be assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a locally fixed complex structure (preferred imaginary unit contained by tangent space at each point of HO) are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

Any hyper-octonion analytic function $OH \rightarrow OH$ defines a function $g : OH \rightarrow SU(3)$ acting as the group of octonion automorphisms leaving a preferred imaginary unit invariant, and g in turn defines a foliation of OH and $H = M^4 \times CP_2$ by space-time surfaces. The selection can be local which means that G_2 appears as a local gauge group.

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by 4-surfaces. Therefore space-time surface can be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of Kähler action.

7. Generalization of ordinary number fields: infinite primes and cognition

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and embedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

1 Introduction

This chapter is devoted to the possible role of infinite primes in TGD and TGD inspired theory of consciousness.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also for their complexifications and one can speak about infinite primes in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors.

1.1 The Notion Of Infinite Prime

The original motivation for the notion of infinite prime came from the first attempts to construct TGD inspired theory of consciousness (around 1995) [K8]. Suppose very naively that the 4-surfaces in a given sector of the "world of classical worlds" (WCW) are labelled by a fixed p-adic prime. The natural expectation is that evolution by quantum jumps means dispersion in the space of these sectors and leads to the increase of the p-adic prime characterizing the Universe. As one moves backwards in subjective time (sequence of quantum jumps) one ends up to the situation in which the prime characterizing the universe was $p = 2$. Should one assume that there was the first quantum jump when everything began? If not, then it would seem that the p-adic prime characterizing the Universe must be infinite. Second problem is that the p-adic length scales are finite and if the size scale of Universe is given by p-adic length scale the Universe has finite sized: this does not make sense in TGD framework. The only way out of the problems is the assumption that the p-adic prime characterizing the entire Universe is literally infinite and that p-adic primes characterizing space-time sheets are finite.

These argument, which are by no means central for the recent view about p-adic primes, motivated the attempt to construct a theory of infinite primes and to extend quantum TGD accordingly. This turns out to be possible. The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. It must be also emphasized that the notion of infinity is relativistic. With respect to the p-adic norm infinite primes have unit norm for all finite and infinite primes so that there is nothing to become scared of!

Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave fuctions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of embedding space and space-time surface are subject to a number theoretic evolution. In philosophical mood one can of course also ask whether there exists a hierarchy of embedding spaces in which the embedding space at the lower level represents something with infinitesimal size in the sense of real topology and whether this hierarchy is accompanied also by a hierarchy of conscious entities.

This picture suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [A1] providing a rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively. Same generalization could make sense for all classical number fields.

1.2 Infinite Primes And Physics In TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1.2.1 Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this way.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and

this cheated me to believe that this supersymmetry is completely absent in TGD Universe. The progress in the understanding of the properties of the modified Dirac action however led to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [**?**].

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [K9] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [K1].

1.2.2 Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

1.2.3 The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K9]. the dark matter hierarchy characterized by increasing values of $\hbar$ [K4]. the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory with finite measurement resolution.

1.2.4 Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. This conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

Quantum classical correspondence requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The Kähler-Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries. The notion of finite measurement resolution allows to deduce much more detailed about this correspondence. In particular, the rational defined by the infinite prime classifies the finite sub-manifold geometry defined by the discretization of the partonic 2-surface implied by the finite measurement resolution. Also a direct correlation between integers defining Planck constant and the "fermionic" part of the infinite prime emerges.

1.3 Infinite Primes, Cognition, And Intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. One can define the notion of prime also for the algebraic extensions of rationals. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum of infinitesimals (real zeros) is replaced by multiplication of real units meaning that the set of real and also more general units becomes infinitely degenerate.

3. Infinite primes form an infinite hierarchy so that the points of space-time and embedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

4. In zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

5. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to

a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

1.4 About Literature

The reader not familiar with the basic algebra of quaternions and octonions is encouraged to study some background material: the home page of Tony Smith provides among other things an excellent introduction to quaternions and octonions [A7]. String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [A2, A5, A4] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [A6]. which is not so elementary as the name would suggest, introduces in enjoyable way the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [A3] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L1].

2 Infinite Primes, Integers, And Rationals

The definition of the infinite integers and rationals is a straightforward procedure and structurally similar to a repeated second quantization of a super-symmetric quantum field theory but including also the number theoretic counterparts of bound states.

2.1 The First Level Of Hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1 + X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{2.1}$$

If P were divisible by finite prime then $P - X = 1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N = P - k$, $k > 1$, are certainly not primes since k can be taken as a factor. The number $P' = P - 2 = -1 + X$ could however be prime. P is certainly not divisible by $P - 2$. It seems that one cannot express P and $P - 2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p \in U} p + q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P - 2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{array}{l} P(\pm, n) = \pm 1 + nX \ , \\ k(p) = 0, 1, \ldots.. \ , \\ n = \prod_p p^{k(p)} \ , \\ X = \prod_p p \ , \end{array} \tag{2.2}$$

where $k(p) \neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good candidates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{array}{l} P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{2.3}$$

Here $x|y$ means "x divides y". To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+, 1, 1, 1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$
\begin{aligned}
&P(\pm, m, n, s|r) = nY^r \pm ms \ , \\
&Y = \frac{X}{s} \ , \\
&m = \prod_{p|s} p^{k(p)} \ , \\
&n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned}
\tag{2.4}
$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$
\begin{aligned}
&N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\
&Y = \frac{X}{s} \ , \\
&n_1m_2 - n_2m_1 = 0 \ .
\end{aligned}
$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteeing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n, m, r)(Y) = nY^r + m$ with integer valued coefficients $(n > 0)$ defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$
\begin{aligned}
&P(\pm, m, n, s|r_1, r_2) = nY^{r_1} \pm ms \ , \\
&m = P_{r_2}(Y)Y + m_0 \ , \\
&Y = \frac{X}{s} \ , \\
&m_0 = \prod_{p|s} p^{k(p)} \ , \\
&n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned}
\tag{2.5}
$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteeing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1: th or r_2: th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p\ mod\ 4 = 3$ condition guaranteeing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm,m,n,s)$. $P(\pm,1,1,1)\ mod\ 4$ is either 3 or 1. The value of $P(\pm,m,n,s)\ mod\ 4$ for odd s on n only and is same for all states containing even/odd number of $p\ mod\ = 3$ excitations. For even s the value of $P(\pm,m,n,s)\ mod\ 4$ depends on m only and is same for all states containing even/odd number of $p\ mod\ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+,m,n,s)$ or $P(-,m,n,s)$ but not both are physically interesting infinite primes ($2m\ mod\ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

2.2 Infinite Primes Form A Hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm,m,n,s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction recipe. In present case "vacuum primes" at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X\prod_{P(\pm,m,n,s)} P(\pm,m,n,s) \ , \\ S = s\prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{2.6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm,m,n,s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type "bosons" respectively. The non-negative integer-valued function $K(P) = K(\pm,m,n,s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type "bosons". More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\begin{array}{l} \frac{P_1(\pm,m_1,n_1,s_1K_1,S_1)}{P_2(\pm,m_2,n_2,s_2,K_2,S_2)} \\ = \frac{n_1s_2}{n_2s_1}\prod_{\pm,m,n,s}(\frac{n}{s})^{K_1^+(\pm,n,m,s)-K_2^+(\pm,n,m,s)} \ . \end{array} \tag{2.7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

2.3 Construction Of Infinite Primes As A Repeated Quantization Of A Super-Symmetric Arithmetic Quantum Field Theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symmetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actually use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the "vacuum" X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ "p-bosons" in mode of type X/s and multiplication of the "vacuum" s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ "p-bosons". in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{2.8}$$

   obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P \ mod\ 4 = 3$ and $P \ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n.

Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,...,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the embedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

2.4 Construction In The Case Of An Arbitrary Commutative Number Field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [K6] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [K6]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [K6]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

2.5 Mapping Of Infinite Primes To Polynomials And Geometric Objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \quad . \tag{2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n: th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n: th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

One can speculate with possible connections to TGD physics. The degree n of the polynomial is its basic characterizer. Infinite primes corresponding to polynomials of degree $n > 1$ should correspond to bound states. On the other hand, the hierarchy of Planck constants suggests strongly the interpretation in terms of gravitational bound states. Could one identify $h_{eff}/h = n$ as the degree of the polynomial characterizing infinite prime?

2.6 How To Order Infinite Primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the "large" and the "small" part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1 S_1}{M_2 S_2} \frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of $M_i S_i$ as ordering criterion. Again the procedure can be repeated if needed.

2.7 What Is The Cardinality Of Infinite Primes At Given Level?

The basic problem is to decide whether Nature allows also integers S, $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0} 2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

2.8 How To Generalize The Concepts Of Infinite Integer, Rational And Real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

2.8.1 Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \qquad (2.10)$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{2.11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{2.12}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \tag{2.13}$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultra-metricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

2.8.2 Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \tag{2.14}$$

2.8.3 Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \tag{2.15}$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \tag{2.16}$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N> \ . \tag{2.17}$$

The natural inner product is

$$\langle X, Y\rangle = x_0 y_0 + \sum_N x_N y_N \ . \tag{2.18}$$

The inner product is well defined if the number of N: s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X + Y = x_0 + y_0 + \sum_N (x_N + y_N) p^{-N} \ , \tag{2.19}$$

The product XY is expressible in the form

$$XY = x_0 y_0 + x_0 Y + X y_0 + \sum_{N_1, N_2} x_{N_1} y_{N_2} p^{-N_1 - N_2} \ , \tag{2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \ \rightarrow \ x_p = \sum_k x_k p^k \ ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \ \rightarrow \ (x_0)_p + \sum_N (x_N)_p p^N \ , \tag{2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \rightarrow x_0 + \sum_N x_N p_2^{-N} . \tag{2.22}$$

The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the embedding space provokes the question whether it might be possible to regard the infinite-dimensional WCW , or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

2.9 Comparison With The Approach Of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement "Set is Many allowing to regard itself as One" really means and to the fact that there is no obvious connection with physics.

The proposed approach is based on the introduction of the concept of prime as a basic concept whereas partial ordering is based on the use of ratios: using these one can recursively define partial ordering and get precise quantitative information based on finite reals. The ordering is only partial and there is infinite number of ratios of infinite integers giving rise to same real unit which in turn leads to the idea about number theoretic anatomy of real point.

The "Set is Many allowing to regard itself as One" is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as "One" and its decomposition to a product of primes corresponds to the set as "Many". The concept of prime, the ultimate "One", has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

3 Can One Generalize The Notion Of Infinite Prime To TheNon-Commutative And Non-Associative Context?

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [K7] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [K7]. Also the hierarchy of infinite primes should generalize as also the representation of infinite primes as polynomials although associativity is expected to pose technical problems.

3.1 Quaternionic And Octonionic Primes And Their Hyper Counterparts

The loss of commutativity and associativity implies that the definitions of quaternionic and octonionic primes are not completely straightforward.

3.1.1 Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space if one uses octonionic representation for the gamma matrices [K7]. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many ways and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio $3:1$. For other Lie-groups this ratio is either 2: 1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

3.1.2 Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

3.1.3 Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic prime has one particualr representative as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$.

Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them if one assumes the entire quaternionic prime as four-momentum: only a system where energy is minimum is possible. The introduction of a preferred hyper-complex plane necessary for several reasons- in particular for the possibility to identify standard model quantum numbers in terms of infinite primes- allows to identify the momentum of particle in the preferred plane as the first two components of the hyper prime in fixed coordinate frame. Note that this leads to a universal spectrum for mass squared.

For time like hyper-primes the momentum is always time like for hyper-primes. In this case it is possible to find a rest frame by applying a hyper-primeness preserving G_2 transformation so that the resulting momentum has no component in the preferred frame. As a matter fact, SU(3) rotation is enough for a suitable choice of SU(3). These transformations form a discrete subgroup of SU(3) since hyper-integer property must be preserved. Massless states correspond to a null norm for the corresponding hyper integer unless one allows also tachyonic hyper primes with minimal representatives $(n_3, n_3 - 1, 0, ...)$, $n_3 = (p-1)/2$. Note that Gaussian primes with $p \ mod 4 \ = 1$ are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$ and would correspond to genuine tachyons. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

The notion of "irreducible" (see Appendix of [K6]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

3.2 Hyper-Octonionic Infinite Primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance. It is however not possible to interpret them as as 8-momenta with mass squared equal to prime. The proper identification of standard model quantum numbers will be discussed later.

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes.

Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

Stronger form of associativity and commutativity is obtained if infinite octonionic/quaternionic primes are just ordinary octonionic/quaternonic primes multiplied with ordinary infinite primes. This option is perhaps the more elegant one. For this option the non-commutativity and non-associativity are concentrated on the finite octonionic/quaternionic prime multiplying the commutative infinite prime. This picture allows also the map of infinite octonionic/quaternionic primes to products of finite octonionic/quaternionic primes and of polynomials.

4 How To Interpret The Infinite Hierarchy Of Infinite Primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

4.1 Infinite Primes And Hierarchy Of Super-Symmetric Arithmetic Quantum Field Theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. It turns also possible to assign to infinite prime space-time surface as a geometric correlate although the original proposal for how to achieve this failed. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-symplectic representations (for instance, ordinary particles correspond to states of this kind).

4.1.1 Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ .$$

2. Form the vacuum states

$$V_{\pm} = X \pm 1 \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

4.1.2 More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed. As found, the conjectured effective 2-dimensionality for hyper-octonionic primes allows the reduction of polynomial representation of hyper-octonionic primes to that for hyper-complex primes. This would be in accordance with the effective 2-dimensionality of the basic objects of quantum TGD.

The physical counterpart of n: th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

4.1.3 How infinite rationals correspond to quantum states and space-time surfaces?

The most promising answer to the question how infinite rationals correspond to space-time surfaces is discussed in detail in the next section. Here it is enough to give only the basic idea.

1. In zero energy ontology hyper-octonionic units (in real sense) defined by ratios of infinite integers have an nterpretation as representations for pairs of positive and negative energy states. Suppose that the quantum number combinations characterizing positive and negative energy quantum states are representable as superpositions of real units defined by ratios of infinite integers at each point of the space-time surface. If this is true, the quantum classical correspondence coded by the measurement interaction term of the Kähler-Dirac action maps the quantum numbers also to space-time geometry and implies a correspondence between infinite rationals and space-time surfaces.

2. The space-time surface associated with the infinite rational is in general not a union of the space-time surfaces associated with the primes composing the integers defining the rational. There the classical description of interactions emerges automatically. The description of classical states in terms of infinite integers would be analogous to the description of many particle states as finite integers in arithmetic quantum field theory. This mapping could in principle make sense both in real and p-adic sectors of WCW .

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime.

4.1.4 What is the interpretation of the higher level infinite primes?

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

4.2 Infinite Primes, The Structure Of Many-Sheeted Space-Time, And The Notion Of Finite Measurement Resolution

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in a rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface. It turns out that the notion of finite measurement resolution is the key concept: infinite prime characterizes angle measurement resolution. This gives a direct connection with the p-adicization program relying also on angle measurement resolution as well as a connection with the hierarchy of Planck constants. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory so that the characterization of the finite measurement resolution, which has been the ugly ducling of theoretical physics transforms to a beautiful swan.

4.2.1 The first intuitions

The concrete prediction of the general vision is that the hierarchy of infinite primes should somehow correspond to the hierarchy of space-time sheets or partonic 2-surfaces if one accepts the effective 2-dimensionality. The challenge is to find space-time counterparts for infinite primes at the lowest level of the hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by flux tubes to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at the lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in

some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on.... A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles. The concrete realization of multi-p p-adicity would be in terms of infinite integers coming as power series $\sum x_n N^n$ and having interpretation as p-adic numbers for any prime dividing N.

2. Effective 2-dimensionality would suggest that the individual p-adic topologies could be assigned with the 2-dimensional partonic surfaces. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. There are however reasons to think that even single partonic 2-surface corresponds to a multi-p p-adic topology.

4.2.2 Do infinite primes code for the finite measurement resolution?

The above describe heuristic picture is not yet satisfactory. In order to proceed, it is good to ask what determines the finite prime or set of them associated with a given partonic 2-surface. It is good to recall first the recent view about the p-adicization program relying crucially on the notion of finite measurement resolution.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M and measurement resolution does not depend on on the value of M. Situation is different if one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer. For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$ if modular arithetics is used to define the measurement resolution. This works at the first level of the hierarchy but not at higher levels. The alternative manner to assign a finite measurement resolution to M/N for given p is as $\Delta\phi = 2\pi|N/M|_p = 2\pi/p^n$. In this case the small fermionic part of the infinite prime would fix the measurement resolution. The argument below shows that only this option works also at the higher levels of hierarchy and is therefore more plausible.

3. p-Adicization conditions in their strong form require that the notion of integration based on harmonic analysis in symmetric spaces makes sense even at the level of partonic 2-surfaces. These conditions are satisfied if the partonic 2-surfaces in a given measurement resolution can be regarded as algebraic continuations of discrete surfaces whose points belong to the discrete variant of the $\delta M^4_\pm \times CP_2$. This condition is extremely powerful since it effectively allows to code the geometry of partonic 2-surfaces by the geometry of finite sub-manifold geometries for a given measurement resolution. This condition assigns the integer N to a given partonic surface and all primes appearing as factors of N define possible effective p-adic topologies assignable to the partonic 2-surface.

How infinite primes could then code for the finite measurement resolution? Can one identify the measurement resolution for $M/N = M/(Rp^n)$ as $\Delta\phi = ((M/R)\ mod\ p^n) \times 2\pi/p^n$ or as $\Delta\phi = 2\pi/p^n$? The following argument allows only the latter option.

1. Suppose that p-adic topology makes sense also for infinite primes and that state function reduction selects power of infinite prime P from the product of lower level infinite primes defining the integer N in M/N. Suppose that the rational defined by infinite integer defines measurement resolution also at the higher levels of the hierarchy.

2. The infinite primes at the first level of hierarchy representing Fock states are in one-one correspondence with finite rationals M/N for which integers M and N can be chosen to characterize the infinite bosonic part and finite fermionic part of the infinite prime. This correspondence makes sense also at higher levels of the hierarchy but M and N are infinite integers. Also other option obtained by exchanging "bosonic" and "fermionic" but later it will be found that only the first identification makes sense.

3. The first guess is that the rational M/N characterizing the infinite prime characterizes the measurement resolution for angles and therefore partially classifies also the finite sub-manifold geometry assignable to the partonic 2-surface. One should define what $M/N = ((M/R)\ mod\ P^n) \times P^{-n}$ is for infinite primes. This would require expression of M/R in modular arithmetics modulo P^n. This does not make sense.

4. For the second option the measurement resolution defined as $\Delta\phi = 2\pi|N/M|_P = 2\pi/P^n$ makes sense. The Fourier basis obtained in this manner would be infinite but all states $exp(ik/P^n)$ would correspond in real sense to real unity unless one allows k to be infinite P-adic integer smaller than P^n and thus expressible as $k = \sum_{m<n} k_m P^m$, where k_m are infinite integers smaller than P. In real sense one obtains all roots $exp(iq2\pi)$ of unity with $q < 1$ rational. For instance, for $n = 1$ one can have $0 < k/P < 1$ for a suitably chosen infinite prime k. Thus one would have essentially continuum theory at higher levels of the hierarchy. The purely fermionic part N of the infinite prime would code for both the number of Fourier components in discretization for each power of prime involved and the ratio characterize the angle resolution.

The proposed relation between infinite prime and finite measurement resolution implies very strong number theoretic selection rules on the reaction vertices.

1. The point is that the vertices of generalized Feynman diagrams correspond to partonic 2-surfaces at which the ends of light-like 3-surfaces describing the orbits of partonic 2-surfaces join together. Suppose that the partonic 2-surfaces appearing a both ends of the propagator lines correspond to same rational as finite sub-manifold geometries. If so, then for a given p-adic effective topology the integers assignable to all lines entering the vertex must contain this p-adic prime as a factor. Particles would correspond to integers and only the particles having common prime factors could appear in the same vertex.

2. In fact, already the work with modelling dark matter [K4] led to ask whether particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions. It also seemed natural to assume that only the space-time sheets containing common primes in this collection can interact. This inspired the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The possibility of multi-p p-adicity raises the question about how to fix the p-adic prime characterizing the mass of the particle. The mass scale of the contribution of a given throat to the mass squared is given by $p^{-n/2}$, where $T = 1/n$ corresponds to the p-adic temperature of throat. Hence the dominating contribution to the mass squared corresponds to the smallest prime power p^n associated with the throats of the particle. This works if the integers characterizing other particles than graviton are divisible by the gravitonic p-adic prime or a product of p-adic primes assignable to graviton. If the smallest power p^n assignable to the graviton is large enough, the mass of graviton is consistent with the empirical bounds on it. The same consideration applies in the case of photons. Recall that the number theoretically very natural condition that in zero energy ontology the number of generalized Feynman graphs contributing to a given process is finite is satisfied if all particles have a non-vanishing but arbitrarily small p-adic thermal mass [K10].

4.2.3 Interpretational problem

The identification of infinite prime as a characterizer of finite measurement resolution looks nice but there is an interpretational problem.

1. The model characterizing the quantum numbers of WCW spinor fields to be discussed in the next section involves a pair of infinite primes P_+ and P_- corresponding to the two vacuum primes $X \pm 1$. Do they correspond to two different measurement resolutions perhaps assignable to CD and CP_2 degrees of freedom?

2. Different measurement resolutions in CD and CP_2 degrees of freedom need not be not a problem as long as one considers only the discrete variants of symmetric spaces involved. What might be a problem is that in the general case the p-adic primes associated with CD and CP_2 degrees of freedom would not be same unless the integers N_+ and N_- are assumed to have have same prime factors (they indeed do if $p^0 = 1$ is formally counted as prime power factors).

3. The idea of assigning different p-adic effective topologies to CD and CP_2 does not look attractive. Both CD and CP_2 and thus also partonic 2-surface could however possess simultaneously both p-adic effective topologies. This kind of option might make sense since the integers representable as infinite powers series of integer N can be regarded as p-adic integers for all prime factors of N. As a matter fact, this kind of multi-p p-adicity could make sense also for the partonic 2-surfaces characterized by a measurement resolution $\Delta\phi = 2\pi M/N$. One would have what might be interpreted as N_+N_--adicity.

4. It will be found that quantum measurement means also the measurement of the p-adic prime selecting same p-adic prime from N_+ and N_-. If $N_\pm$ is divisible only by $p^0 = 1$, the corresponding angle measurement resolution is trivial. From the point of view of consciousness state function reduction selects also the p-adic prime characterizing the cognitive representation which is very natural since quantum superpositions of different p-adic topologies are not natural physically.

4.3 How The Hierarchy Of Planck Constants Could Relate To Infinite Primes And P-Adic Hierarchy?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K9], the dark matter hierarchy characterized by increasing values of $\hbar$ [K3, K2], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations give hopes about developing a more quantitative vision about the relationship between these hierarchies, in particular between the hierarchy of infinite primes, p-adic length scale hierarchy, and the hierarchy if Planck constants.

If infinite primes code for the hierarchy of measurement resolutions, the correlations between the p-adic hierarchy and the hierarchy of Planck constants indeed suggest themselves and allow also to select between two interpretations for the fact that two infinite primes N_+ and N_- are needed to characterize elementary particles (see the next section).

Recall that the hierarchy of Planck constants in the most general situation corresponds to a replacement M^4 and CP_2 factors of the embedding space with singular coverings and factor spaces. The condition that Planck constant is integer valued allows only singular coverings characterized by two integers n_a *resp.* n_b assignable to CD *resp.* CP_2. This condition also guarantees that a given value of Planck constant corresponds to only a finite number of pages of the "Big Book" and therefore looks rather attractive mathematically. This option also forces evolution as a dispersion to the pages of the books characterized by increasing values of Planck constant.

Concerning the correspondence between the hierarchy of Planck constants and p-adic length scale hierarchy there seems to be only single working option. The following assumptions make precise the relationship between finite measurement resolution, infinite primes and hierarchy of Planck constants.

1. Measurement resolution CD *resp.* CP_2 degrees of freedom is assumed to correspond to the rational M_+/N_+ *resp.* M_-/N_-. $N_\pm$ is identified as the integer assigned to the fermionic part of the infinite integer..

2. One must always fix the consideration to a fixed p-adic prime. This process could be regarded as analogous to fixing the quantization axes and p would also characterize the p-adic cognitive space-time sheets involved. The p-adic prime is therefore same for CD and CP_2 degrees of freedom as required by internal consistency.

3. The relationship to the hierarchy of Planck constants is fixed by the identifications $n_a = n_+(p)$ and $n_b = n_-(p)$ so that the number of sheets of the covering equals to the number of bosons in the fermionic mode p of the quantum state defined by infinite prime.

4. A physically attractive hypothesis is that number theoretical bosons *resp.* fermions correspond to WCW orbital *resp.* spin degrees of freedom. The first ones correspond to the symplectic algebra of WCW and the latter one to purely fermionic degrees of freedom.

Consider now the basic consequences of these assumptions from the point of view of physics and cognition.

1. Finite measurement resolution reduces for a given value of p to

$$\Delta\phi = \frac{2\pi}{p^{n_\pm(p)+1}} = \frac{2\pi}{p^{n_{a/b}}} \quad ,$$

   where $n_\pm(p) = n_{a/b} - 1$ is the number of bosons in the mode p in the fermionic part of the state. The number theoretical fermions and bosons and also their probably existing physical counterparts are necessary for a non-trivial angle measurement resolution. The value of Planck constant given by

$$\frac{\hbar}{\hbar_0} = n_a n_b = (n_+(p)+1) \times (n_-(p)+1)$$

   tells the total number of bosons added to the fermionic mode p assigned to the infinite prime.

2. The presence of $\hbar > \hbar_0$ partonic 2-surfaces is absolutely essential for a Universe able to measure its own state. This is in accordance with the interpretation of hierarchy of Planck constants in TGD inspired theory of consciousness. One can also say that $\hbar = 0$ sector does not allow cognition at all since $N_\pm = 1$ holds true. For given p $\hbar = n_a n_b = 0$ means that given fermionic prime corresponds to a fermion in the Dirac sea meaning $n_\pm(p) = -1$. Kicking out of fermions from Direac sea makes possible cognition. For purely bosonic vacuum primes one has $\hbar = 0$ meaning trivial measurement resolution so that the physics is purely classical and would correspond to the purely bosonic sector of the quantum TGD.

3. For $\hbar = \hbar_0$ the number of bosons in the fermionic state vanishes and the general expression for the measurement resolution reduces to $\Delta\phi = 2\pi/p$. When one adds $n_\pm(p)$ bosons to the fermionic part of the infinite prime, the measurement resolution increases from $\Delta\phi = 2\pi/p$ to $\Delta\phi = 2\pi/p^{n_\pm(p)+1}$. Adding a sheet to the covering means addition of a number theoretic boson to the fermionic part of infinite prime. The presence of both number theoretic bosons and fermions with the values of p-adic prime $p_1 \neq p$ does not affect the measurement resolution $\Delta\phi = 2\pi/p^n$ for a given prime p.

4. The resolutions in CD and CP_2 degrees of freedom correspond to the same value of the p-adic prime p so that one has dicretizations based on $\Delta\phi = 2\pi/p^{n_a}$ in CD degrees of freedom and $\Delta\phi = 2\pi/p^{n_b}$ in CP_2 degrees of freedom. The finite sub-manifold geometries make sense in this case and since the effective p-adic topology is same, the continuation to continuous p-adic partonic 2-surface is possible.

p-Adic thermodynamics involves the p-adic temperature $T = 1/n$ as basic parameter and the p-adic mass scale of the particle comes as $p^{-(n+1)/2}$. The natural question is whether one could assume the relation $T_{\pm} = 1/(n_{\pm}(p)+1)$ between p-adic temperature and infinite prime and thus the relations $T_a = 1/n_a(p)$ and $T_b = 1/n_b(p)$. This identification is not consistent with the recent physical interpretation of the p-adic thermodynamics nor with the view about dark matter hierarchy and must be given up.

1. The minimal non-trivial measurement resolution with $n_i = 1$ and $\hbar = \hbar_0$ corresponds to the p-adic temperature $T_i = 1$. p-Adic mass calculations indeed predict $T = 1$ for fermions for $\hbar = \hbar_0$. In the case of gauge bosons $T \geq 2$ is favored so that gauge bosons would be dark. This would require that gauge bosons propagate along dark pages of the Big Book and become "visible" before entering to the interaction vertex.
2. p-Adic thermodynamics also assumes same p-adic temperature in CD and CP_2 degrees of freedom but the proposed identification allows also different temperatures. In principle the separation of the super-conformal degrees of freedom of CD and CP_2 might allow different p-adic temperatures. This would assign to different p-adic mass scales to the particles and the larger mass scale should give the dominant contribution.
3. For dark particles the p-adic mass scale would be by a factor $1/\sqrt{p}^{n_i(p)-1}$ lower than for ordinary particles. This is in conflict with the assumption that the mass of the particle does not depend on $\hbar$. This prediction would kill completely the recent vision about the dark matter.

5 How Infinite Primes Could Correspond To Quantum States And Space-time Surfaces?

The hierarchy of infinite primes is in one-one correspondence with a hierarchy of second quantizations of an arithmetic quantum field theory. The additive quantum number in question is energy like quantity for ordinary primes and given by the logarithm of prime whereas p-adic length scale hypothesis suggests that the conserved quantity is proportional to the inverse of prime or its square root. For infinite primes at the first level of hierarchy these quantum numbers label single particles states having interpretation as ordinary elementary particles. For octonionic and hyper-octonionic primes the quantum number is analogous to a momentum with 8 components. The question is whether these number theoretic quantum numbers could have interpretation as genuine quantum numbers. Quantum classical correspondence raises another question. Is it possible to label space-time surfaces by infinite primes? Could this correspondence be even one-to-one?

I have considered these questions already more than decade ago. The discussion at that time was necessarily highly speculative and just a mathematical exercise. After that time however a lot of progress has taken place in quantum TGD and it is highly interaction to see what comes out from the interaction of the notion of infinite prime with the notions of zero energy ontology and generalized embedding space, and with the recent vision about how Chern-Simons Dirac term in the Kähler-Dirac action allows to code information about quantum numbers to the space-time geometry. The possibility of this coding allows to simplify the discussion dramatically. If one could map infinite hyper-octonionic or hyper-quaternionic primes to quantum numbers of the standard model naturally, then the their map of to the geometry of space-time surfaces would realize the coding of space-time surfaces by infinite primes (and more generally by integers and rationals). Also a detailed realization of number theoretic Brahman=Atman identity would emerge as an outcome.

5.1 A Brief Summary About Various Moduli Spaces And Their Symmetries

It is good to sum up the number theoretic symmetries before trying to construct an overall view about the situation. Several kinds of number theoretical symmetry groups are involved corresponding to symmetries in the moduli spaces of hyper-octonionic and hyper-quaternionic structures, symmetries mapping hyper-octonionic primes to hyper-octonionic primes, and translations acting in the space of causal diamonds (CDs) and shifting. The moduli space for CDs labeled by pairs of its tips that its pairs of points of $M^4 \times CP_2$ is also in important role.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit. $SO(7,1)$ acts as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for hyper-quaternionic structures.

2. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

3. Color group $SU(3)$ is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. For given hyper-octonionic prime one can identify a subgroup of SU(3) generating a finite set of hyper-octonionic primes for it at sphere S^7. This suggests wave function at the orbit of given hyper-octonionic prime in turn generalizing to wave functions in the space of infinite primes.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

These symmetries deserve a more detailed discussion.

1. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures. $SO(7)$ respects the choice of the real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as sub-structures of hyper-octonionic structure. The choice of global hyper-octonionic structures involves also a choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of CD. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture. The choice of both tips of CD reduces $SO(1,3)$ to $SO(3)$.

2. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

3. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit meaning a choice of preferred hyper-complex plane $M^4 \subset M^4$. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement in number theoretical and as it turns also in physical sense. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

5.2 Associativity And Commutativity Or Only Their Quantum Variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Hyper-quaternionicity formulated in terms of the Kähler-Dirac gamma matrices defined by Kähler action fixes classical space-time dynamics and a very beautiful algebra formulation of quantum TGD in terms of hyper-octonionic local Clifford algebra of embedding space emerges. There is no need for the use of hyper-octonion real analytic maps although one cannot exclude the possibility that they might be involved with the construction of hyper-quaternionic space-time surfaces.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to complex rational infinite primes. Since one can decompose complex rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative and commutative. In case of associativity this means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

5.3 How Space-Time Geometry Could Be Coded By Infinite Primes?

Second key question is whether space-time geometry could be characterized in terms of infinite primes (and integers and rationals in the most general case) and how this is achieved. The question is how the quantum states consisting of fundamental fermions serving as building bricks of elementary particles could be coded by infinite quaternionic integeres to which one can assign ordinary finite quaternionic primes.

The basic idea is roughly that at the first level of the hierarchy the finite primes appearing as building blocks of infinite prime correspond to structures formed by pairs or wormhole contacts assigned with elementary particles.

1. The partonic orbits defined by wormhole throats could be characterized by finite primes specifying the preferred p-adic topology assignable to the p-adic "cognitive representation" of the throat.

2. One could assign hyper-quaternionic integer to the real particle as its four-momentum. In this case the mass shell condition would fix the hyper-quaternionic integer to a high extent. All discrete Lorentz boosts of the particle state taking hyper-quaternionic integers to hyper-quaternionic integers would correspond to the same p-adic integer (prime) defined by the length of the Lorentz boosted hyper-quaternionic integer. The p-adic prime characterizing virtual particle would be one of the primes appearing in the factorization of this integer to a product of powers of prime, most naturally the one whose power is largest.

   Note that p-adic length scale hypothesis suggests that the p-adic primes near powers of two are favored for on mass shell particles and perhaps also for the virtual particles.

3. For fundamental fermions associated with boundaries of string world sheets and appearing as building bricks of particles the masses would vanish on mass shell so that the hyper-quaternionic integer would in this case have vanishing norm.

   The virtual four-momentum assigned to a virtual fermion line as a generalized eigenvalue of Chern-Simons Dirac operator would correspond to hyper-quaternionic integer. In this case p-adic prime would be defined as for physical particles and would depend on the mass of the virtual particle. If the integration over virtual momenta by residue calculus effectively leads to an integral over on mass shell massless virtual momenta with non-physical spinor helicities then also virtual fundamental fermions would correspond to zero norm hyper-quaternionic integers.

4. The correlation between particle's four-momentum and the p-adic prime characterizing corresponding cognitive representation would be in accordance with quantum classical correspondence.

5. The hyperquaternionic primes appearing as largest factors in the factorizion of hyper-quaternionic integers assignable with physical particles could be interpreted as building bricks of an infinite hyperquaternionic prime characterizing the many-particle state and at least the boundaries of string world sheets. The idea that p-adic space-time surfaces defined "cognitive representations" as p-adic chart maps of real space-time surfaces and vice versa (as the TGD based definition of p-adic manifolds assumes) suggests that the p-adic primes in question characterize also space-time regions rather than only the boundaries of string world sheets.

A couple of comments about this speculation are in order.

1. ZEO implies a hierarchy of CDs within CDs and this hierarchy as well as the hierarchy of space-time sheets corresponds naturally to the hierarchy of infinite primes. One can assign standard model quantum numbers to various partonic 2-surfaces with positive and negative energy parts of the quantum state assignable to the light-like boundaries of CD. Also infinite integers and rationals are possible and the inverses of infinite primes would naturally correspond to elementary particles with negative energy. The condition that zero energy state has vanishing net quantum numbers implies that the ratio of infinite integers assignable to zero energy state equals to real unit in real sense and has has vanishing total quantum numbers.

2. Neither quantum numbers nor infinite primes coding them cannot characterize the partonic 2-surface itself completely since they say nothing about the deformation of the space-time surface but only about labels characterizing the WCW spinor field. Also the topology of partonic 2-surface fails to be coded. Quantum classical correspondence however suggests that this correspondence could be possible in a weaker sense. In the Gaussian approximation for functional integral over the world of classical worlds space-time surface and thus the collection of partonic 2-surfaces is effectively replaced with the one corresponding to the maximum of Kähler function, and in this sense one-one correspondence is possible unless the situation is non-perturbative. In this case the physics implied by the hierarchy of Planck constants could however guarantee uniqueness.

   One of the basic ideas behind the identification of the dark matter as phases with non-standard value of Planck constant is that when perturbative description of the system fails, a phase transition increasing the value of Planck constant takes place and makes perturbative description possible. Geometrically this phase transition means a leakage to another sector of the embedding space realized as a book like structure with pages partially labeled by the values of Planck constant. Anyonic phases and fractionization of quantum numbers is one possible outcome of this phase transition. An interesting question is what the fractionization of the quantum numbers means number theoretically.

6 Infinite Primes And Mathematical Consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

6.1 Algebraic Brahman=Atman Identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes, which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_{\pm} = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_{\pm} \ mod \ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labeled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.

2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.

3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense.

6.1.1 Number theoretic anatomy of space-time point

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of embedding space force the conclusion that WCW spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of embedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the WCW spinor fields regarded as wave functions in the set of embedding space points which are equivalent in real sense. Embedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single embedding space point.

To realize this picture would require that WCW spinor fields and perhaps even WCW allow a mapping to the number theoretic anatomies of space-time point. In finite-dimension Euclidian spaces momentum space labelling plane wavs is dual to the space. One could hope that also now the "orbital" quantum numbers of WCW spinor fields could code for WCW in given measurement resolution. The construction of the previous sections realize the mapping of the quantum states defined by WCW spinors fields assignable to given CD to wave function in the space of hyper-octonionic units. These wave functions can be also regarded as linear combinations of these units if the coefficients are complex numbers formed using the commuting imaginary unit of complexified octonions so that the Hilbert space like structure in question would have purely number theoretic meaning. The rationals defined by infinite primes characterize also measurement resolution and classify the finite sub-manifold geometries associated with partonic two-surfaces. At higher levels one has rationals defined by ratios of infinite integers and one can ask whether this interpretation generalizes.

Note that one must distinguish between two kinds of hyper-octonionic units.

1. Already in the case of complex numbers one has rational complex units defined in terms of Pythagorean triangle and their products generate infinite dimensional space. The hyper-octonionic units defined as ratios U of infinite integers and suggested to provide a representation of WCW spinor fields correspond to these. The powers U^m define roots of unity which can be regarded analogous to $exp(i2\pi x)$, where x is not rational but the exponent itself is complex rational.

2. Besides this there are roots of unity which are in general algebraic complex numbers. These roots of unit correspond to phases $exp(i2\pi M/N)$, where M/N is ratio of real infinite integers and i is the commuting hyper-octonionic imaginary unit. These real infinite integers can be assigned to hyper-octonionic integers by replacing everywhere finite hyper-octonionic primes with their norm which is ordinary prime. By the previous considerations only the phases $exp(i2\pi M/P^n)$ make sense p-adically for infinite primes P.

6.2 Leaving The World Of Finite Reals And Ending Up To The Ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals at the first level of hierarchy. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. WCW is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of embedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced WCW consisting of the maxima of Kähler function to the anatomy of space-time point. Also WCW spinors and perhaps also the modes of WCW spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of embedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even "simultaneous" time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just "epsilons" if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

6.3 Infinite Primes And Mystic World View

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{aligned}
&P = nY^{r_1} + mS \ , \quad r = 1, 2, ... \\
&m = m_0 + P_{r_2}(Y) \ , \\
&Y = \frac{X}{S} \ , \\
&S = \prod_i P_i \ .
\end{aligned}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

   If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

   The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightenment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1!$), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the WCW . In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level “beings” (one might call them Angels, Gods, etc...).

6.4 Infinite Primes And Evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labeling the WCW sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first WCW spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [K5]. Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two ways.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by.... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

   (a) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

   (b) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

   The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle: the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

7 Does The Notion Of Infinite-P P-Adicity Make Sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing the tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The following list of questions is rather natural with the background provided by the p-adic physics.

1. Can one generalize the notion of p-adic norm and p-adic number field to include infinite primes? Could one define the counterpart of p-adic topology for literally infinite values of p? Does the topology R_P for infinite values of P approximate or is it equivalent with real topology as p-adic topology at the limit of infinite p is assumed to do (at least in the sense that p-adic variants of Diophantine equations at this limit correspond to ordinary Diophantine equations)? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale.

2. Canonical identification maps p-adic numbers of unit norm to real numbers in the range $[0,p]$. Does the canonical identification map the p-adic numbers R_P associated with infinite prime to reals? Could the number fields R_P provide alternative formulations/generalizations of the non-standard analysis based on the hyper-real numbers of Robinson [A1] ?

3. The notion of finite measurement resolution for angle variables given naturally as a hierarchy $2\pi/p^n$ of resolutions for a given p-adic prime defining aa hierarchy of algebraic extension of p-adic numbers is central in the attempts to formulate p-adic variants of quantum TGD and fuse them with real number based quantum TGD [K6] . If p is replaced with an infinite prime, the angular resolution becomes ideal and the roots of unity $exp(2\pi m/p^n)$ are replaced with real units unless also the integer m is replaced with an infinite integer M so that the ratio M/P^n is finite rational number. Could this approach be regarded as alternative for real number based notion of phase angle?

The consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p. Using infinite primes one might obtain the real theory in this approximation.

The question discussed in this section is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can formally introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

7.1 Does Infinite-P P-Adicity Reduce To Q-Adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \quad (7.1)$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement "infinite integer modulo P" means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime. The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarchy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1+x_1P+..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1+x_1P+...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the "finite field" defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1+y_1P+...)$$

and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the lowest order term in the polymomial expansion. Thus the norm would be infinite for $n < 0$, equal to one for $n = 0$ and vanish for $n > 0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

For the lowest level infinite primes the natural choice of a would be the rational number defined by it so that infinite-P p-adicity would indeed correspond to q-adicity meaning that number field property is lost.

7.2 Q-Adic Topology Determined By Infinite Prime As A Local Topology Of WCW?

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the WCW .

This view conforms also with the idea of algebraic holography. The sub-spaces of WCW can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between WCW and number theoretic anatomy of point of the embedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to WCW integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteeing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

7.3 The Interpretation Of The Discrete Topology Determined By Infinite Prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naïvely generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naïve sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy of preferred extremals: one can add to any preferred extremal a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also Kähler-Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since $p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

Acknowledgements

I want to thank Tony Smith and Carlos Castro for useful discussions and references related to quaternions and octonions.

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