Are higher structures needed in the categorification of TGD?

M. Pitkänen,

June 19, 2019
Email: matpitka6@gmail.com.
http://tgdtheory.com/public_html/.
Recent postal address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland.

Contents

Abstract

The notion of higher structures promoted by John Baez looks very promising notion in the attempts to understand various structures like quantum algebras and Yangians in TGD framework. The stimulus for this article came from the nice explanations of the notion of higher structure by Urs Screiber. The basic idea is simple: replace "=" as a blackbox with an operational definition with a proof for $A = B$. This proof is called homotopy generalizing homotopy in topological sense. n-structure emerges when one realizes that also the homotopy is defined only up to homotopy in turn defined only up...

In TGD framework the notion of measurement resolution defines in a natural manner various kinds of "="s and this gives rise to resolution hierarchies. Hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adeles in adelic TGD and corresponding hierarchy of Galois groups represented geometrically, hierarchy of infinite primes, self hierarchy, etc...

In this article the idea of n-structure is studied in more detail. A rather radical idea is a formulation of quantum TGD using only cognitive representations consisting of points of space-time surface with imbedding space coordinates in extension of rationals defining the level of adelic hierarchy. One would use only these discrete points sets and Galois groups. Everything would reduce to number theoretic discretization at space-time level perhaps reducing to that at partonic 2-surfaces with points of cognitive representation carrying fermion quantum numbers.

Even the "world of classical worlds " (WCW) would discretize: cognitive representation would define the coordinates of WCW point. One would obtain cognitive representations of scattering amplitudes using a fusion category assignable to the representations of Galois groups: something diametrically opposite to the immense complexity of the WCW but perhaps consistent with it. Also a generalization of McKay's correspondence suggests itself: only those irreps of the Lie group associated with Kac-Moody algebra that remain irreps when reduced to a subgroup defined by a Galois group of Lie type are allowed as ground states. Also the relation to number theoretic Langlands correspondence is very interesting.

1 Introduction

I encountered a very interesting work by Urs Schreiber related to so called higher structures and realized that these structures are part of the mathematical language for formulating quantum TGD in terms of Yangians and quantum algebras in a more general manner.

1.1 Higher structures and categorification of physics

What theoretical physicist Urs Screiber calls "higher structures" are closely related to the categorification program of physics. Baez, David Corfield and Urs Schreiber founded a group blog n-Category Cafe about higher category theory and its applications. John Baez is a mathematical physicists well-known from is pre-blog "This Week's Finds" (see `http://tinyurl.com/yddcabfl`) explaining notions of mathematical physics.

Higher structures or n-structures involve "higher" variants of various mathematical structures such as groups, algebras, homotopy theory, and also category theory (see `http://tinyurl.com/ydz9mbtp`. One can assign a higher structure to practically anything. Typically one loosens some conditions on the structure such as commutativity or associativity: a good example is the product for octonionic units which is associative only apart from sign factors [K12]. Braid groups and fusion algebras [L3], which seem to play crucial role in TGD can be seen as higher structures.

The key idea is simple: replace "=" with homotopy understood in much more general sense than in topology and identified as the procedure proving $A = B$! Physicist would call this operationalism. I would like a more concrete interpretation: "=" is replaced with "=" in a given measurement resolution. Even homotopies can be defined only modulo homotopies of homotopies - that is within measurement resolution - and one obtains a hierarchy of homotopies and at the highest level coherence conditions state that one has "=" almost in the good old sense. This kind of hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adels in adelic TGD, hierarchy of infinite primes, self hierarchy, etc...

1.2 Evolution of Schreiber's ideas

One of Schreiber's articles in Physics Forum articles has title "*Why higher category theory in physics?*" (see http://tinyurl.com/ydcylrun) telling his personal history concerning the notion of higher category theory. Supersymmetric quantum mechanics and string theory/M-theory are strongly involved with his story.

1.2.1 Wheeler's superspace and its deformations as starting point

Schreiber started with super variant of Wheeler's super-space. Intriguingly, also the "world of classical worlds" (WCW) of TGD [K4, K2, K18] emerged as a counterpart of superspace of Wheeler in which the generalization of super-symmetries is geometrized in terms of spinor structure of WCW expressible in terms of fermionic oscillator operators so that there is something common at least.

Screiber consider deformation theory of this structure. Deformations appear also in the construction of various quantum structures such as quantum groups and Yangians. Both quantum groups characterized by quantum phase, which is root of unity, and Yangians ideal for reduction of many-particle states and their interactions to kinematics seem to be the most important from the TGD point of view [L3].

These deformations are often called "quantizations" but this nomenclature is to my opinion misleading. In TGD framework the basic starting point is "*Do not quantize*" meaning the reduction of the entire quantum theory to classical physics at the level of WCW: modes of a formally classical WCW spinor fields correspond to the states of the Universe.

This does not however prevent the appearance of the deformations of basic structures also in TGD framework and they might be the needed mathematical tool to describe the notions of finite measurement resolution and cognitive resolution appearing in the adelic version of TGD. I proposed more than decade ago that inclusions of hyperfinite factors of II_1 (HFFs) [K14, K3] might provide a natural description of finite measurement resolution: the action of included factor would generate states equivalent under the measurement resolution used.

1.2.2 The description of non-point-like objects in terms of higher structures

Schreiber ends up with the notion of higher gauge field by considering the space of closed loops in 4-D target space [B5]. At the level of target space the loop space connection (1-form in loop space) corresponds to 2-form at the level of target space. At space-time level 1- form A defines gauge potentials in ordinary gauge theory and non-abelian 2-form B as its generalization with corresponding higher gauge field identified as 3-form $F = dB$.

The idea is that the values of 2-form B are defined for a string world sheet connecting two string configuration just like the values of 1-form are defined for a world-line connecting two positions of a point-like particle. The new element is that the ordinary curvature form does not anymore satisfy the usual Bianchi identities stating that magnetic monopole currents are vanishing (see http://tinyurl.com/ya3ur2ad).

It however turns out that one has $B = DA = F$ (D denotes covariant derivative) so that B is flat by the usual Bianchi-identities implying $dB = 0$ so that higher gauge field vanishes. B also turns out to be Abelian. In the Abelian case the value of 2-form would be magnetic flux depending only on the boundary of string world sheet. By $dB = 0$ gauge fields in loop space would vanish and only topology of field configurations would make itself manifest as for locally trivial gauge potentials in topological quantum field theories (TQFT): a generalization of Aharonov-Bohm effect would be in question. Schreiber calls this "*fake flatness condition*". This could be seen as an unsatisfactory outcome since dynamics would reduce to topological dynamics.

The assumption that loop space gauge fields reduce to those in target space could be argued to be non-realistic in TGD framework . For instance, high mass excitations of theories of extended structures like strings would be lost. In the case of loop spaces there is also problem with general coordinate invariance (GCI): one would like to have 2-D GCI assignable to string world sheets. In TGD the realization that one must have 4-D GCI for 3-D fundamental objects was a breakthrough, which occurred around 1990 about 12 years after the discovery of the basic idea of TGD and led to the discovery of WCW Kähler geometry and to "Do not quantize".

1.2.3 Understanding "fake flatness" condition

Schreiber tells how he encountered the article of John Baez titled "*Higher Yang-Mills Theory*" [B4] (see `http://tinyurl.com/yagkqsut`) based on the notion of 2-category and was surprised to find that also now the "*fake flatness condition*" emerged.

Schreiber concludes that the "*fake flatness condition*" results from "a kind of choice of coordinate composition": non-Abelian higher gauge field would reduce to Abelian gauge field over a background of ordinary non-Abelian gauge fields. Schreiber describes several string theory related examples involving branes and introduces connection with modern mathematics. Since branes in the stringy sense are not relevant to TGD and I do not know much about them, I will not discuss these here.

However, dimensional hierarchies formed by fermions located to points at partonic 2-surfaces, their world lines at 3-D light-like orbits of partons, strings and string world sheets as their orbits, and space-time surfaces as 4-D orbits of 3-surfaces definitely define a TGD analog for the brane hierarchy of string models. It is not yet completely clear whether strong form of holography (SH) implies that string world sheets and strings provide dual descriptions of 4-D physics or whether one could regard all levels of this hierarchy independent to some degree at least [L1].

Since the motion of measurement resolution is fundamental in TGD [K14, K3], it is interesting to see whether n-structures could emerge naturally also in TGD framework. There is also second aspect involved: various hierarchies appearing in TGD have basically the structure of abstraction hierarchy of statements about statements and higher structures seem to define just this kind of hierarchies. Of course, human mind - at least my mind - is in grave difficulties already with few lowest levels but here category theory and its computerization might come into a rescue.

1.3 What higher structures are?

Schreiber describes in very elegant and comprehensible manner the notion of higher structures (see `http://tinyurl.com/ydfspcld`). This description is a real gem for a physicists frustrated to the impenetrable formula jungle of the usual mathematical prose. Just the basic ideas and the reader can start to think using his/her own brains. The basic ideas ideas are very simple and general. Even if one were not enthusiastic about the notion of higher gauge field, the notion of higher structure is extremely attractive concerning the mathematical realization of the notion of finite measurement resolution.

1. The idea is to reconsider the meaning of "=". Usually it is understood as equivalence: $A = B$ if A and B belong to same equivalence class defined by equivalence relation. The idea is to replace "=" with its operational definition, with the proof of equivalence. This could be seen as operationalism of physics applied to mathematics. Schreiber calls this proof homotopy identified as a generalization of a map $f_t\colon S \to X$ depending on parameter $t \in [0, 1]$ transforming two objects of a topological space X to each other in continuous manner: $f_0(S)$ is the initial object and $f_1(S)$ is the final object. Now homotopy would be much more general.

2. One can also improve the precision of "=" meaning that equivalence classes decompose to smaller ones and equivalent homotopies decompose to subclasses of equivalent homotopies related by homotopies. One might say that "=" is deconstructed to more precise "=". Physicist would see this as a partial opening of a black box by improving the measurement resolution. This gives rise to n-variants of various algebraic structures.

3. This hierarchy would have a finite number of levels. At highest level the accuracy would be maximal and "=" would have almost its usual meaning. This idea is formulated in terms of coherence conditions. Braiding involving R-matrix represents one example: permutations are replaced by braidings and permutation group is lifted to braid group but associativity still holds true for Yang-Baxter equation (YBE). Second example is 2-group for which associativity holds true only modulo homotopy so that $(x \circ y) \circ z$ is related to $x \circ (y \circ z)$ by homotopy $a_{x,y,z}$ depending on x, y, z and called an associator. For 2-group the composite homotopy $((w \circ x) \circ y) \circ z \to w \circ (x \circ (y \circ z))$ is however unique albeit non-trivial.

   This gives rise to the so called pentagon identity encountered also in the theory of quantum groups and Yangians. The outcome is that all homotopies associated with re-bracketings of an algebraic expression are identical. One can define in similar manner n-group and formally even infinity-group.

1.4 Possible applications of higher structures to TGD

Before listing some of the applications of higher structures imaginable in TGD framework, let us summarize the basic principles.

1. Physics as WCW geometry [K10, K4, K2, K18] having super-symplectic algebra (SSA) and partonic super-conformal algebra (PSCA) as fundamental symmetries involving a generalization of ordinary conformal invariance to that for light-like 3-surfaces defined by the boundary of CD and by the light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian.

2. Physics as generalized number theory [K11] [L4] leading to the notion of adelic physics with a hierarchy of adeles defined by the extensions of rationals.

3. In adelic physics finite resolutions for sensory and cognitive representations (see the glossary of Appendix) could would characterize "=". Hierarchies of resolutions meaning hierarchies of n-structures rather than single n-structure would give inclusion hierarchies for HFFs, SSA, and PSCA, and extensions of rationals characterized by Galois groups with order identifiable as $h_{eff}/h = n$ and ramified primes of extension defining candidates for preferred p-adic primes.

   Finite measurement resolution defined by SSA and its isomorphic sub-algebra acting as pure gauge algebra would reduce SSA to finite-dimensional SKMA. WCW could become effectively a coset space of Kac-Moody group or of even Lie group associated with it. Same would take place for PSCA. This would give rise to n-structures. Quantum groups and Yangians would indeed represent examples of n-structures.

   In TGD the "conformal weight" of Yangian however corresponds to the number of partonic surfaces - parton number - whereas for quantum groups and Kac-Moody algebras it is analogous to harmonic oscillator quantum number n, which however has also interpretation as boson number. Maybe this co-incidence involves something much deeper and relates to quantum classical correspondence (QCC) remaining rather mysterious in quantum field theories (QFTs).

4. An even more radical reduction of degrees of freedom can be imagined. Cognitive representations could replace space-time surfaces with discrete structures and points of WCW could have cognitive representations as disretized WCW coordinates.

5. Categorification requires morphisms and homomorphisms mapping group to sub-group having normal sub-group defining the resolution as kernel would define "resolution morphisms". This normal sub-group principle would apply quite generally. One expects that the representations of the groups involved are those for quantum groups with quantum phase q equal to a root of unity.

Some examples helps to make this more concrete.

1.4.1 Scattering amplitudes as computations

The deterministic time devolution connecting two field patterns could define analog of homotopy in generalized sense. In TGD framework space-time surface (preferred extremals) having 3-D space-like surfaces at the opposite boundaries of causal diamond (CD) could therefore define analog of homotopy.

1. Preferred extremal defines a topological scattering diagram in which 3-vertices of Feynman diagram are replaced with partonic 2-surfaces at which the ends of light-like orbits of partonic 2-surfaces meet and fermions moving along lines defined by string world sheets scatter classically, and are redistributed between partonic orbits [K23, K21, K26]. Also braidings

and reconnections of strings are possible. It is important to notice that one does not sum over these topological diagrams. They are more like possible classical backgrounds.

The conjecture is that scattering diagrams are analogous to algebraic computations so that one can find the shortest computation represented by a tree diagram. Homotopy in the roughest sense could mean identification of topological scattering diagrams connecting two states at boundaries of CD and differing by addition of topological loops. The functional integral in WCW is proposed to trivialize in the sense that loop corrections vanish as a manifestation of quantum criticality of Kähler coupling strength and one obtains an exponent of Kähler function which however cancels in scattering amplitudes if only single maximum of Kähler function contributes.

2. In the optimal situation one could eliminate all loops of these diagrams and also move line ends along the lines of diagrams to get tree diagrams as representations of scattering diagrams. Similar conditions hold for fusion algebras. This might however hold true only in the minimal resolution. In an improved measurement resolution the diagrams could become more complex. For instance, one might obtain genuine topological loops.

3. The diagrams and state spaces with different measurement resolutions could be related by Hilbert space *isometries* but would not be unitarily equivalent: Hilbert space isometries are also defined by entanglement in tensor nets [K17]. This would give an n-levelled hierarchy of higher structures (rather than single n-structure!) and at the highest level with best resolution one would have coherence rules. Generalized fusion algebras would partially realize this vision. In improved measurement resolution the diagrams would not be identical anymore and equivalence class would decompose to smaller equivalence classes. This brings in mind renormalization group equations with cutoff.

4. Intuitively the improvement of the accuracy corresponds to addition of sub-CDs of CDs and smaller space-time sheets glued to the existing space-time sheets.

1.4.2 Zero energy ontology (ZEO)

In ZEO [K16] "=" could mean the equivalence of two zero energy states indistinguishable in given measurement resolution. Could one say that the 3-surfaces at the ends of space-time surface are equivalent in the sense that they are connected by preferred extremal and have thus same total Noether charges, or that entangled many-fermion states at the boundaries of CD correspond to quantal logical equivalences (fermionic oscillator algebra defines a quantum Boolean algebra)?

In the case of zero energy states "=" could tolerate a modification of zero energy state by zero energy state in smaller scale analogous to a quantum fluctuation in quantum field theories (QFTs). One could add to a zero energy state for given CD zero energy states associated with smaller CDs within it.

In TGD inspired theory of consciousness [L6] sub-CDs are correlates for the perceptive fields of conscious entities and the states associated with sub-CDs would correspond to sub-selves of self defining its mental images. Also this could give rise to hierarchies of n-structures with n characterizing the number of CDs with varying sizes. An interesting proposal is the distance between the tips of CD is integer multiple of CP_2 for number theoretic reasons. Primes and primes near powers of 2 are suggested by p-adic length scale hypothesis [K6, K7, K8] [L4].

1.4.3 "World of classical worlds" (WCW)

At the level of "world of classical worlds" (WCW) "=" could have both classical meaning and meaning in terms of quantum state defining the measurement resolution. At the level of WCW geometry n-levelled hierarchies formed by the isomorphic sub-algebras of SSA and PSCA are excellent candidates for n-structures. The sub-SCA or sub-PSCA would define the measurement resolution. The smaller the sub-SSA or sub-PSCA, the better the resolution.

This could correspond to a hierarchy of inclusions of HFFs [K14, K3] to which one can assign ADE SKMA by McKay correspondence or its generalization allowing also other Lie groups suggested by the hierarchy of extensions of rationals with Galois groups that are groups of Lie type.

The conjecture generalizing McKay correspondence is that the Galois group *Gal* is representable as a subgroup of G in the case that it is of Lie type.

An attractive idea is that WCW is effectively reduced to a finite-dimensional coset space of the Kac-Moody group defined by the gauge conditions. Number theoretic universality requires that these parameters belong to the extension of rationals considered so that the Kac-Moody group G is discretized and also homotopies are discretized. SH raises the hope that it is enough to consider string world sheets with parameters (WCW coordinates) in the extension of rationals.

One can define quite concretely the action of elements of homotopy groups of Kac-Moody Lie groups G on space-time surfaces as induced action changing the parameters characterizing the space-time surface. $n+1$-dimensional homotopy would be 1-dimensional homotopy of n-dimensional homotopy. Also the spheres defining homotopies could be discretized so that the coordinates of its points would belong to the extension of rationals.

These kind of homotopy sequences could define analogs of Berry phases (see `http://tinyurl.com/yd4agwnt`) in Kac-Moody group. Could gauge theory for Kac-Moody group give an approximate description of the dynamical degrees of freedom besides the standard model degrees of freedom? This need not be a good idea. It is better to base the considerations of the physical picture provided by TGD. I have however discussed the TGD analog of the *fake flatness condition* in the Appendix.

1.4.4 Adelic physics

Also number theoretical meaning is possible for "=". It is good to start with an objection against adelic physics. The original belief was that adelic physics forces preferred coordinates. Indeed, the property of belonging to an extension of rationals does not conform with general coordinate invariance (GCI). Coordinate choice however matters cognitively as any mathematical physicist knows! One can therefore introduce preferred coordinates at the imbedding space level as cognitively optimal coordinates: they are dictated to a high degree by the isometries of H. One can use a sub-set of these coordinates also for space-time surfaces, string world sheets, and partonic 2-surfaces.

1. Space-time surfaces can be regarded as multi-sheeted Galois coverings of a representative sheet [L4]. Minimal resolution means that quantum state is Galois singlet. Improving resolution means requiring that singlet property holds true only for normal sub-group H of Galois group *Gal* and states belong to the representations of Gal/H. Maximal resolution would mean that states are representations of the entire *Gal*. The hierarchy of normal subgroups of *Gal* would define a resolution hierarchy and perhaps an analog of n-structure. $h_{eff}/h = n$ hypothesis suggests hierarchies of Galois groups with dimensions n_i dividing n_{i+1}. The number of extensions in the hierarchy would characterize n-structure.

2. The increase of the complexity for the extension of rationals would bring new points in the *cognitive representations* defined by the points of the space-time surface with imbedding space coordinates in the extension of rationals used (see the glossary in Appendix). Also the size of the *Gal* would increase and higher-D representations would become possible. The value of $h_{eff}/h = n$ identifiable as dimension of *Gal* would increase. The cognitive representation would become more precise and the topology of the space-time surface would become more complex.

3. In adelic TGD "=" could have meaning at the level of cognitive representations. One could go really radical and ask whether discrete cognitive representations replacing space-time surfaces with the set of points with H-coordinates in an extension of rationals (see the glossary in Appendix) defining the adele should provide the fundamental data and that all group representations involved should be realized as representations of *Gal*. This might apply in cognitive sector.

   This would also replace space-time surfaces as points of WCW with their cognitive representations defining their WCW coordinates! All finite groups can appear as Galois groups for some number field. Whether this is case when one restricts the consideration to the extensions of rationals, is not known. Most finite groups are groups of Lie type and thus representable as rational points of some Lie group. Note that rational point can also mean rational point in extension of rationals as ratio of corresponding algebraic integers identifiable as roots of monic polynomials $P_n(x) = x^n + ...$ having rational coefficients.

4. By SH space-time surface would in information theoretic sense effectively reduce to string world sheets and even discrete set of points with H-coordinates in extension of rationals. These points could even belong to the partonic 2-surface at the ends of strings at ends of CD carrying fermions and the partonic 2-surfaces defining topological vertices. If only this data is available, the WCW coordinates of space-time surface would reduce to these points of $H = M^4 \times CP_2$ and to the direction angles of strings emerging from these points and connecting them to the corresponding points at other partonic 2-surfaces besides Gal identifiable as sub-group of Lie group G of some Kac-Moody group! Not all pairs $Gal - G$ are possible.

5. Could these data be enough to describe mathematically what one knows about space-time surface as point of WCW and the physics? One could indeed deduce $h_{eff}/h = n$ as the order of Gal and preferred p-adic primes as ramified primes of extension. The Galois representations acting on the covering defining space-time surface or string world sheets should be identifiable as representations of physical states. There is even number theoretical vision about coupling constant evolution relying on zeros of Riemann zeta [K20],

6. This sounds fine but one must notice that there is also the global information about the conformal moduli of partonic 2-surfaces and the elementary particle vacuum functionals defined in this moduli space [K1] explain family replication phenomenon. There is also information about moduli of CDs. Also the excitations of SKMA representations with higher conformal weights are present and play a crucial role in p-adic thermodynamics predicting particle masses [K6]. It is far from clear whether the approach involving only cognitive representation is able to describe them.

To help the reader I have included a vocabulary at the end of the article and include here a list of the abbreviations used in the text.

General abbreviations: Quantum field theory (QFT); Topological quantum field theory (TQFT); Hyper-finite factor of type II_1 (HFF); General coordinate invariance (GCI); Equivalence Principle (EP).

TGD related abbreviations: Topological Geometrodynamics (TGD); General Relativity Theory (GRT); Zero energy ontology (ZEO); Strong form of holography (SH); Strong form of general coordinate invariance (SGCI); Quantum classical correspondence (QCC); Negentropy Maximization Principle (NMP); Negentropic entanglement (NE); Causal diamond (CD); Super-symplectic algebra (SSA); Partonic superconformal algebra (PSCA); Super Virasoro algebra (SVA); Kac-Moody algebra (KMA); Super-Kac-Moody algebra (SKMA);

2 TGD very briefly

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K10] and physics as generalized number theory [K11]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

2.1 World of classical worlds (WCW)

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K10] and physics as generalized number theory [K11]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

2.1.1 Construction of WCW geometry briefly

In the following the vision about physics in terms of classical physics of spinor fields of WCW is briefly summarized.

1. The idea is to geometrize not only the classical physics in terms of geometry of space-time surfaces but also quantum physics in terms of WCW [K18]. Quantum states of the Universe would be modes of classical spinor fields in WCW and there would be no quantization. One must construct Kähler metric and Kähler form of WCW: in complex coordinates they differ by a multiplicative imaginary unit. Kähler geometry makes possible to geometrize hermitian conjugation fundamental for quantum theory.

2. One manner to build WCW metric this is via the construction of gamma matrices of WCW in terms of second quantized oscillator operators for fermions described by induced spinor fields at space-time surfaces. By strong form of holography this would reduce to the construction of second quantized induced spinor fields at string world sheets. The anti-commutators of of WCW gamma matrices expressible in terms of oscillator operators would define WCW metric with maximal isometry group (SCA) [K15, K18].

3. Second manner to achieve the geometrization is to construct Kähler metric and Kähler form directly [K4, K2, K18]. The idea is to induce WCW geometry from the Kähler form J of the imbedding space $H = M^4 \times CP_2$. The mere existence of the Riemann connection forces a maximal group of isometries. In fact, already in the case of loop space the Kähler geometry is essentially unique.

   The original construction used only the Kähler form of CP_2. The twistor lift of TGD [K26] forces to endow also M^4 with the Minkowskian analog of Kähler form involving complex and hypercomplex part and the sum of the two Kähler forms can be used to define what might be called flux Hamiltonians. They would define the isometries of WCW as symplectic transformations. What was surprising and also somewhat frustrating was that what I called almost 2-dimensionality of 3-surfaces emerges from the condition of general coordinate invariance and absence of dimensional parameters apart from the size scale of CP_2.

   In the recent formulation this corresponds to SH: 2-D string world sheets and 2-D partonic 2-surfaces would contain data allowing to construct space-time surfaces as preferred extremals. In adelic physics also the specification of points of space-time surface belonging to extension of rationals defining the adele would be needed. There are several options to consider but the general idea is clear.

   SH is analogous to a construction of analytic function of 2-complex from its real values at 2-D surface and the analogy at the level of twistor lift is holomorphy as generalization of holomorphy of solutions gauge fields in the twistor approach of Penrose. Also quaternionic analyticity [K23] is suggestive and might mean even stronger form of holography in which 1-D data allow to construct space-time surfaces as preferred extremals and quantum states.

   I have proposed formulas for the Kähler form of WCW in terms of flux Hamiltonians but the construction as anti-commutators of gamma matrices is the more convincing definition. Fermions and second quantize induced spinor fields could be an absolutely essential part of WCW geometry.

4. WCW allows as infinitesimal isometries huge super-symplectic algebra (SSA) [K4, K2] acting on space-like 3-surfaces at the ends of space-time surfaces inside causal diamond (CD) and also generalization of Kac-Moody and conformal symmetries acting on the 3-D light-like orbits of partonic 2-surfaces (partonic super-conformal algebra (PSCA)). These symmetry algebras have a fractal structure containing a hierarchy of sub-algebras isomorphic to the full algebra. Even ordinary conformal algebra with non-negative conformal weights has similar fractal structure as also Yangian. In fact, quantum algebras are formulated in terms of these half algebras.

   The proposal is that sub-algebra of SSA (with non-negative conformal weights) and isomorphic to entire SSA and its commutator with the full algebra annihilate the physical states. What remains seems to be finite-D Kac-Moody algebra as an effective "coset" algebra obtained. Note that the resulting normal sub-group is actually quantum group.

   There is direct analogy with the decomposition of a group Gal to a product of sub-group and normal sub-group H. If the normal sub-group H acts trivially on the representation the representation of Gal reduces to that of the group Gal/H. Now one works at Lie algebra level: *Gal* is replaced with SSA and H with its sub-algebra with conformal weights multiples of those for SSA.

2.1.2 Super-symplectic conformal weights, zeros of Riemann zeta, and quantum phases?

In [K20] I have considered the possibility that the generators of super-symplectic algebra could correspond to zeros $h = 1/2 + iy$ of zeta. The hypothesis has several variants.

1. The simplest variant is that the non-trivial zeros of zeta are labelling the generators of SSA associated with Hamiltonians proportional to the functions $f(r_M)$ of the light-like radial coordinate of light-cone boundary as $f(r_M) = (r_M/_0)^h \equiv exp(hu)$, $u = log(r_M/r_0)$, $h = -1/2 + iy$. For infinitely large size of CD the plane waves are orthogonal but for finite-sized CD orthogonality is lost. Orthogonality requires periodic boundary conditions and these are simultanwously possible only for a finite number of zeros of zeta.

2. One could modify the hypothesis by allowing superpositions of zeros of zeta but with a subtraction of half integer to make the real part of ih equal to $1/2$ so that one obtains an analog of plane-wave when using $u = log(r_M/r_0)$ as a radial coordinate. Equivalently, one can take dr_M/r_M out as integration measure and assume $h = iy$ plus the condition that the Riemannian plane waves are orthogonal and satisfy periodic boundary conditions for the allowed zeros $z = 1/2 + iy$.

3. Periodic boundary conditions can be satisfied for given zero of zeta if the condition $r_{max}/r_{min} = p^n$ holds true and the additional conjecture that given non-trivial zeros of zeta correspond to prime $p(y)$ and p^{iy} is a root of unity. Given basis of $f(r_M)$ would correspond to n-ary p-adic length scales and also the size scales of CDs would correspond to powers of p-adic primes. This conjecture is rather attractive physically and I have not been able to prove it wrong.

   One can associate to given zero $z = 1/2 + iy$ single and only single prime $p(y)$ by demanding that $p^{iy} = exp(i2\pi q)$, $q = m/n$ rational, implying $log(p)y = 2\pi q$. If there were two primes p_1 and p_2 of this kind, one one ends up with contradiction $p_1^m = p_2^n$ for some integers m and n.

   One could however associate several zeros $y_i(p)$ to the same prime p as discussed in [K20]. If $N = \prod_i n_i$ is the smallest common denominator of q_i allowed conformal weights would be superpositions $ih = iN \sum n_i y_i(p)$ and conformal weights would form higher dimensional lattice rather than 1-D lattice as usually. If only single prime $p(y)$ can be associated to given y, then the original hypothesis identifying $h = 1/2 + iy$ as conformal weight would be natural.

4. The understanding of the p-adic length scale hypothesis is far from complete and one can ask whether preferred p-adic primes near powers of 2 and possibly also other small primes could be primes for which there are several roots $y_i(p)$.

2.2 Strong form of holography (SH)

There are several reasons why string world sheets and partonic 2-surfaces should code for physics. One reason for SH comes from $M^8 - H$ correspondence [K19]. Second motivation comes from the condition that spinor modes at string world sheets are eigenstates of em charge [K15]. The third reason could come the requirement that the notion of commutative quantum sub-manifold [A1] is equivalent with its number theoretic variant.

2.2.1 SH and $M^8 - H$ correspondence

The strongest form of $M^8 - H$ correspondence [K12, K19, K26] assumes that the 4-surfaces $X^4 \subset M^8$ have fixed $M^2 \subset M^4 \subset M^8$ as part of tangent space. A weaker form states that these 2-D subspaces M^2 define an integrable distribution and therefore 2-D surface in M^4. This condition guarantees that the quaternionic (associative) tangent space of X^4 is parameterized by a point of CP_2 so that the map of X^4 to a 4-surface in $M^4 \times CP_2$ is possible. One can consider also co-associative space-time surfaces having associative normal spaces. m Note that $M^8 - H$ [K12, K19] correspondence respects commutativity and quaternionic property by definition since it maps space-time surfaces having quaternionic tangent space having fixed M^2 as sub-set of tangent space.

What could be the relationship between SH and $M^8 - H$ correspondence? Number theoretic vision suggests rather obvious conjectures.

1. Could the tangent spaces of string world sheets in H be commutative in the sense of complexified octonions and therefore be hyper-complex in Minkowskian regions. By $M^8 - H$ duality the commutative sub-manifolds would correspond to those of octonionic M^8 and finding of these could be the first challenge. The co-commutative manifolds in quaternionic X^4 would have commutative normal spaces. Could they correspond to partonic 2-surfaces?

2. There is however a delicacy involved. Could world sheets and partonic 2-surfaces correspond to hyper-complex and co-hyper-complex sub-manifolds of space-time surface X^4 identifiable as quaternionic surface in octonionic M^8 mappable to similar surfaces in H. Or could their M^4 (CP_2) projections define hypercomplex (co-hypercomplex) 2-manifolds?

3. Could co-commutativity condition for a foliation by partonic 2-surfaces select preferred string world sheets as normal spaces integrable to 2-surfaces identifiable as string world sheets? Note that induced gauge field on 2-surface is always Abelian so that QFT and number theory based views about commutativity co-incide.

   Preferred choices for these 2-surfaces would serve as natural representatives for the equivalence classes of string world sheets and partonic 2-surfaces with fermions at the boundaries of string world sheets serving as markers for the representatives? The end points of the string orbits would belong to extension of rationals or even correspond to singular points at which the different sheets co-incide and have rational coordinates: this possibility was considered in [L7].

Real curves correspond to the lowest level of the dimensional hierarchy of continuous surfaces. Could string world lines along light-like partonic orbits correspond to real sub-manifolds of octonionic M^8 mapped to $M^4 \times CP_2$ by $M^8 - H$ correspondence and carrying fermion number?

What about the set of points with coordinates in the extension of rationals? Do all these points carry fermion number? If so they must correspond to the edges of the boundaries of string world sheets at partonic 2-surfaces at the boundaries of CD or edges at the partonic 2-surfaces defining generalized vertices to which sub-CDs could be assigned.

2.2.2 Well-definedness of em charge forces 2-D fundamental objects

The proposal has been that the representative string world sheets should have vanishing induced W fields so that induced spinors could have well-defined em and Z^0 charges and partonic 2-surfaces would correspond to the ends of 3-D boundaries between Euclidian and Minkowskian space-time regions [K15, K18].

As a matter of fact, the projections of electroweak gauge fields to 2-D surfaces are always Abelian and by using a suitable $SU(2)_L \times U(1)$ rotation one can always find a gauge in which the induced W fields and even Z^0 field vanish. The highly non-trivial conclusion is that string world sheets as fundamental dynamical objects coding 4-D physics by SH would guarantee well-definedness of em charge as fermionic quantum number. Also the projections of all classical color gauge fields, whose components are proportional to $H^A J$, where H^A is color Hamiltonian and J is Kähler form of CP_2, are Abelian and in suitable gauge correspond to hypercharge and isospin.

One can imagine a foliation of space-time surfaces by string world sheets and partonic 2-surfaces. Could there be a $U(1)$ gauge invariance allowing to chose partonic 2-surfaces and string world sheets arbitrarily? If so, the assignment of the partonic 2-surfaces to the light-like boundaries between Minkowskian and Euclidian space-time regions would be only one - albeit very convenient - choice. I have proposed that this choice is equivalent with the choice of complex coordinates of WCW. The change of complex coordinates would introduce a $U(1)$ transformation of Kähler function of WCW adding to it a real part of holomorphic function and of Kähler gauge potential leaving the Kähler form and Kähler metric of WCW invariant.

2.2.3 String world sheets as sub-manifolds of quantum spaces for which commuting sub-set of coordinates are diagonalized?

The third notion of commutativity relates to the notion of non-commutative geometry. Unfortunately, I do not know much about non-commutative geometry.

1. Should one follow Connes [A1] and replace string world sheets with non-commutative geometries with quantum dimension identifiable as fractal dimension. I must admit that I have felt aversion towards non-commutative geometries. For linear structures such as spinors the quantum Clifford algebra looks natural as a "coset space" obtained by taking the orbits of included factor as elements of quantum Clifford algebra. The application of this idea to string world sheets does not look attractive to me.

2. The basic reason for my aversion is that non-commutative quantum coordinates lead to problems with general coordinate invariance (GCI). There is however a possible loophole here. One can approach the situation from two angles: number theoretically and from the point view of non-commutative space. Commutativity could mean two things: number theoretic commutativity and commutativity of quantum coordinates for H seen as observables. Could these two meanings be equivalent as quantum classical correspondence (QCC) encourages to think?

   Could the discreteness for cognitive representations correspond to a discretization of the eigenvalue spectrum of the coordinates as quantum operators? The choice of the coefficient number field for Hilbert space as extension of rationals would automatically imply this and resolve the problems related to continuous spectra.

   Quantum variant of string world sheet could correspond to a quantization using a subset of imbedding space coordinates as quantum commutative coordinates as coordinates for string world sheet. H-coordinates for string world sheet would correspond to eigenvalues of commuting quantum coordinates.

The above three views about SH suggests that Abelianity at the fundamental level is unavoidable because basic observable objects are 2-dimensional. This would correspond $A = J = -B = 0$ for non-Abelian gauge fields reducing to Abelian ones in Schreiber's approach. Also Schreiber finds that with suitable choice of coordinates this holds true always. In TGD this choice would correspond to gauge choice in which all induced gauge fields are Abelian (see Appendix).

Ordinary twistorialization maps points of M^4 to bi-spinors allowing quantum variants. Could twistorialization of M^4 and CP_2 allow something analogous?

3 The notion of finite measurement resolution

Finite measurement resolution [K14, K3] is central in TGD. It has several interpretations and the challenge is to unify the mutually consistent views.

3.1 Inclusions of HFFs, finite measurement resolution and quantum dimensions

Concerning measurement resolution the first proposal was that the inclusions of HFFs chacterize it.

1. The key idea is simple. Yangians and/or quantum algebras associated with the dynamical SKMAs defined by pairs of SSA and its isomorphic sub-algebra acting as pure gauge transformations are characterized by quantum phases [L3] characterizing also inclusions of HFFs [K14, K3]. Quantum parameter would characterize the measurement resolution.

   The Lie group characterizing SKMA would be replaced by its quantum counterpart. Quantum groups involve quantum parameter $q \in C$ involved also with n-structures. This parameter - in particular its phase- should belong to the extension of rationals considered. Notions like braiding making sense for 2-D structures are crucial. Remarkably, the representation theory for quantum groups with q different from a root of unity does not differ from that for ordinary groups. For the roots of unity the situation is different.

2. The levels in the hierarchy of inclusions for HFFs [K14] are labelled by integer $n \in [3, \infty)$ or equivalenly by quantum phases $q = exp(i\pi/n)$ and quantum dimension is given by $d_q = 4cos^2(\pi/n)$. $n = 3$ gives $d = 2$ that is ideal SH with minimal measurement resolution. For instance, in extension of rationals only phases, which are powers of $exp(i\pi/3)$ are represented p-adically so that angle measurement is very imprecise. The hierarchy would correspond to an increasing measurement resolution and at the level $n \to \infty$ one would have $d_q \to 4$. Could the interpretation be that one sees space-time as 4-dimensional? This strongly suggests that the hierarchy of Lie groups characterizing SKMAs are characterized by the same quantum phase as inclusions of HFFs.

How does quantal dimension show itself at space-time level?

1. Could SH reduce the 4-surfaces to effectively fractal objects with quantum dimension d_q? Could one speak of quantum variant of SH perhaps describe finite measurement resolution. In adelic picture this limit could correspond to an extension of rational consists of algebraic numbers extended by all rational powers of e. How much does this limit deviate from real numbers?

2. McKay correspondence (see `http://tinyurl.com/z48d92t`) states that the hierarchy of finite sub-groups of $SU(2)$ corresponds to the hierarchy ADE Kac-Moody algebras in the following sense. The so called McKay graph codes for the information about the multiplicities of the tensor products of given representation of finite group (spin 1/2 doublet) - obviously one can assign McKay graph to any Galois group. McKay correspondence says that the McKay graph for the so called canonical representation of finite sub-group of $SU(2)$ co-incides with the Dynkin diagram for ADE type Kac-Moody algebra.

3. A physically attractive idea is that these algebras correspond to a hierarchy of reduced SSAs and PSCAs defined by the gauge conditions of SSA and PSCA. The breaking of maximal effective gauge symmetry characterizing measurement resolution to isomorphic sub-algebra would bring in additional degrees of freedom increasing the quantum dimension of string world sheets from the minimal value $d_q = 2$.

   My naive physical intuition suggests that McKay correspondence generalizes to a much wider class of Galois groups identifiable as finite groups of Lie type identifiable as sub-groups of Lie groups (for the periodic table of finite groups see (see `http://tinyurl.com/y75r68hp`)). In general, the irreducible representation (irrep) of group is reducible representation of subgroup. The rule could be that the representations of the quantum Lie groups *allowed* as ground states of SKMA representations are *irreducible* also as representations of Galois group in case that it is Lie-type subgroup.

What about the concrete geometric interpretation of d_q? Two interpretations, which do not exclude each other, suggest themselves.

1. A very naive idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. The idea about fractal string world sheets does not however conform with the fact that preferred extremals must be rather smooth.

   String world sheets could be however locally smooth if they define an analog of discretization for the space-time surface. At the limit $d_q \to 4$ string world sheets would fill space-time surface. Analogously, strings (string orbits) would fill the space-like 3-surfaces at the boundaries of CD (the light-like 3-surfaces connecting the partonic 2-surfaces at boundaries of CD). The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution at the level of physics. For anyonic systems [K9] one indeed would have have large number of fermions at 2-D surfaces.

2. An alternative idea is that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. A precise mathematical representation of ignorance can be extremely successful trick as ordinary thermodynamics and also p-adic thermodynamics for particle masses [K6] demonstrate!

3.2 Three options for the identification of quantum dimension

The quantum dimension would increase as the measurement accuracy increases but what quantum dimension of string world sheets could mean at space-time level? Identification of quantum dimension as fractal dimension could be the answer but how could one concretely define this notion? Could one find an elegant formulation for the fractality at space-time level.

3.2.1 Option I

One could argue that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. One would give up the attempts to represent quantum superposition of space-time surfaces with single classical surface. This option would use only the discrete cognitive representations (see the glossary in Appendix).

1. This would mean a radical simplification and could make sense for cognitive representations. String world sheet would be replaced by this discrete cognitive representation and one should be able to deduce its quantum dimension. *Gal* acts on this representation.

2. Could one imagine q-variants of the representations of *Gal* defining also representations of the Lie group defining KMA? If one can imbed *Gal* to Lie-group as discrete sub-group then the q-representation of the Lie-group would define a q-representation of discrete group and one might be able to talk about q-Galois groups.

3. On the other hand, the condition that these representations restricted to representations of Galois group remain irreducible poses similar condition. Are these two criteria equivalent? Could this allow to identify the value of root of unity associated with given Galois group and corresponding Lie group defining SKMA in case that it contains representations that remain irreps of Galois group? If so, the notion of quantum group would follow from adelic physics in a natural manner.

   This would allow to assign quantum dimension to the discretized string world sheet without clumsy fractal constructions at space-time level involving a lot of redundant information. The really nice thing would be that one would use only the information defining the cognitive representations and the fact that one does not know about the rest. Just as in thermodynamics, things would become extremely simple!

4. One might argue that giving just discrete points at partonic 2-surfaces gives very little information. If one however assumes that also the functions characterizing space-time surfaces as points of sub-WCW involved are constructed from rational polynomials with roots in the extension of rationals used, the situation improves dramatically.

3.2.2 Option II

A very naive idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. Smooth strings would fill the space-like 3-surfaces at the boundaries of CD and light-like 3-surface connecting the partonic 2-surfaces at boundaries of CD. The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution. For anyonic systems one indeed would have have large number of fermions at 2-D surfaces.

This option would be based on fractal dimension of some kind. Most naturally the fractal dimension would be that of space-time surface discretized using string world sheets and possibly also partonic 2-surface instead of points. It is however difficult to imagine a practical realization for fractal dimension in this sense.

1. Assume reference string world sheets in the minimal resolution defined by an extension of rationals with total area S_0. Study the total area S associated with string world sheets as function of the extension of rationals.

2. As the size of the extension grows, new points of extension emerge at partonic 2-surfaces and therefore also new string world sheets and the total area of string worlds sheets increases. Twistor lift suggests that one can take the area S_1 defined by Planck length squared and the area S_2 of CP_2 geodesic sphere as units. Suppose that one has $S/S_0 = (S_1/S_2)^d$, where d depends on the extension and equals to $d = 0$ for rationals, holds true. Could $d+2$ define the fractal dimension equal to d_q for Jones inclusions in the range $[2,4)$? If the proposed notion of quantum Galois group makes sense this could be the case.

One must admit that the hopes of proving this picture works in practice are rather meager. Too much redundant information is involved.

3.2.3 Option III

One can also imagine an approach quantum dimension identifying quantum dimension as fractal dimension for space-time surface. If SH makes sense, one can consider the possibility that this dimension determined by the geometry of space-time surface as Riemann manifold has fractal dimension equal to the fractal dimension of string world sheets as sub-manifold.

1. The spectral dimension of classical geometry is discussed in `http://tinyurl.com/yadcmjd6`). One considers heat equation describing essentially random walk in a given metric and constructs so called heat kernel as a solution of the heat equation. The Laplacian depends on metric only - now the induced metric. The trace of heat kernel characterizes the probability to return to the original position. The derivative of the logarithm of the heat trace with respect to the logarithm of fictive time coordinate gives time dependent spectral dimension, which for short times approaches to topological dimension and for flat space equals to it always. For long times the dimension is smaller than the topological dimension due to curvature effects and SH raises the hope that this dimension corresponds to the fractal dimension of string world sheets identified as quantum dimension.

2. This approach can be criticized for the introduction of fictive time coordinate. Furthermore, Laplacian would be replaced with d'Alembertian in Minkowskian regions so that one cannot speak about diffusion anymore. Could one replace the heat equation with 4-D spinor d'Alembertian or modified Dirac operator so that also the induced gauge fields would appear in the equation? Artificial time coordinate would be replaced with some time coordinate for M^4 - light-cone proper time is the most natural choice. The probability would be defined as modulus squared for the fermionic propagator integrated over space-time surface.

The problem is that this approach is rather formal and might be of little practical value.

3.3 n-structures and adelic physics

TGD involves several concepts, which could relate to n-structures. The notion of finite measurement resolution realized in terms of HFFs is the oldest notion [K14, K3]. Adelic physics suggests that the measurement resolution could be realized in terms of a hierarchy of extensions of rationals [L4]. The parameters characterizing space-time surfaces and by SH the string world sheets would belong to the extension. Also the points of space-time surface in the extension would be data coding for the preferred extremals. The reconnection points and intersection points would belong to the extension [L3]. n-structures relate closely to the notion of non-commutative space and strings world sheets could be such. Also the role of classical number fields - in particular $M^8 - H$ correspondence suggest the same. The challenge is to develop a coherent view about all these structures.

1. There should be also a connection with the adelic view. In this picture string world sheets and points of space-time surface with coordinates in the extension of rationals defining the adele code for the data for preferred extremals and quantum states. What these points are - could they correspond to points of partonic 2-surfaces carrying fermions or could the correspond also to the points in the interior of space-time surface is not clear. The larger the extension of rationals, the larger the number of these points, and the better the resolution and the larger the deviation of SH from ideal. The hierarchy of Galois groups of extension of rationals should relate closely to the inclusion hierarchies.

2. Galois extension with given Galois group Gal allows hierarchy of intermediate extensions defining inclusion sequence for Galois groups. Besides inclusion homomorphisms there exists homomorphisms from Galois group Gal with order $h_{eff}/h = n$ to its sub-groups $H \subset Gal$ with order $h_{eff}/h = m < n$ dividing n. If it exists the sub-group mapped to identity element is normal sub-group H for which right and left cosets gH and Hg are identical. These homomorphisms to sub-groups identify the sheets of Galois covering of the space-time surface transformed to each other by H and thus define different number theoretical resolutions: measurement resolution would have precise geometric meaning. This would mean looking states with $h_{eff}/h = n$ in poorer resolution defined by $h_{eff}/h = m < n$.

   These arrows would define "resolution morphisms" in category theoretic description. Also the analogy with the homotopies of n-structures is obvious. There would be a finite number of normal sub-groups with order dividing n for given higher structure. Quantum phase equal to root of unity ($q = exp(i2\pi/k)$) could appear in these representations and distinguish them from ordinary group representations.

3.4 Could normal sub-groups of symplectic group and of Galois groups correspond to each other?

Measurement resolution realized in terms of various inclusion is the key principle of quantum TGD. There is an analogy between the hierarchies of Galois groups, of fractal sub-algebras of SSA, and of inclusions of HFFs. The inclusion hierarchies of isomorphic sub-algebras of SSA and of Galois groups for sequences of extensions of extensions should define hierarchies for measurement resolution. Also the inclusion hierarchies of HFFs are proposed to define hierarcies of measurement resolutions. How closely are these hierarchies related and could the notion of measurement resolution allow to gain new insights about these hierarchies and even about the mathematics needed to realize them?

1. As noticed, SSA and its isomorphic sub-algebras are in a relation analogous to the between normal sub-group H of group Gal (analog of isomorphic sub-algebra) and the group G/H. One can assign to given Galois extension a hierarchy of intermediate extensions such that one proceeds from given number field (say rationals) to its extension step by step. The Galois groups H for given extension is normal sub-group of the Galois group of its extension. Hence Gal/H is a group. The physical interpretation is following. Finite measurement resolution defined by the condition that H acts trivially on the representations of Gal implies that they are representations of Gal/H. Thus Gal/H is completely analogous to the Kac-Moody type algebra conjecture to result from the analogous pair for SSA.

2. How does this relate to McKay correspondence stating that inclusions of HFFs correspond to finite discrete sub-groups of $SU(2)$ acting as isometries of regular n-polygons and Platonic solids correspond to Dynkin diagrams of ADE type SKMAs determined by ADE Lie group G. Could one identify the discrete groups as Galois groups represented geometrically as sub-groups of $SU(2)$ and perhaps also those of corresponding Lie group? Could the representations of Galois group correspond to a sub-set of representations of G defining ground states of Kac-Moody representations. This might be possible. The sub-groups of $SU(2)$ can however correspond only to a very small fraction of Galois groups.

Can one imagine a generalization of ADE correspondence? What would be required that the representations of Galois groups relate in some natural manner to the representations as Kac-Moody groups.

3.4.1 Some basic facts about Galois groups and finite groups

Some basic facts about Galois groups mus be listed before continuing. Any finite group can appear as a Galois group for an extension of some number field. It is known whether this is true for rationals (see `http://tinyurl.com/hus4zso`).

Simple groups appear as building bricks of finite groups and are rather well understood. One can even speak about periodic table for simple finite groups (see `http://tinyurl.com/y75r68hp`). Finite groups can be regarded as a sub-group of permutation group S_n for some n. They can be classified to cyclic, alternating , and Lie type groups. Note that alternating group A_n is the subgroup of permutation group S_n that consists of even permutations. There are also 26 sporadic groups and Tits group.

Most simple finite groups are groups of Lie type that is rational sub-groups of Lie groups. Rational means ordinary rational numbers or their extension. The groups of Lie type (see `http://tinyurl.com/k4hrqr6`) can be characterized by the analogs of Dynkin diagrams characterizing Lie algebras. For finite groups of Lie type the McKay correspondence could generalize.

3.4.2 Representations of Lie groups defining Kac-Moody ground states as irreps of Galois group?

The goal is to generalize the McKay correspondence. Consider extension of rationals with Galois group Gal. The ground staes of KMA representations are irreps of the Lie group G defining KMA. Could the allow ground states for given Gal be irreps of also Gal?

This constraint would determine which group representations are possible as ground states of SKMA representations for a given Gal. The better the resolution the larger the dimensions of the allowed representations would be for given G. This would apply both to the representations of the SKMA associated with dynamical symmetries and maybe also those associated with the standard model symmetries. The idea would be quantum classical correspondence (QCC) space-time sheets as coverings would realize the ground states of SKMA representations assignable to the various SKMAs.

This option could also generalize the McKay correspondence since one can assign to finite groups of Lie type an analog of Dynkin diagram (see `http://tinyurl.com/k4hrqr6`). For Galois groups, which are discrete finite groups of $SU(2)$ the hypothesis would state that the Kac-Moody algebra has same Dynkin diagram as the finite group in question.

To get some perspective one can ask what kind of algebraic extensions one can assign to ADE groups appearing in the McKay correspondence? One can get some idea about this by studying the geometry of Platonic solids (see `http://tinyurl.com/p4rwc76`). Also the geometry of Dynkin diagrams telling about the geometry of root system gives some idea about the extension involved.

1. Platonic solids have p vertices and q faces. One has $\{p,q\} \in \{\{3,3\},\{4,3\},\{3,4\},\{5,3\},\{3,5\}\}$. Tetrahedron is self-dual (see `http://tinyurl.com/qdl4sss` object whereas cube and octahedron and also dodecahedron and icosahedron are duals of each other. From the table of `http://tinyurl.com/p4rwc76` one finds that the cosines and sines for the angles between the vectors for the vertices of tetrahedron, cube, and octahedron are rational numbers. For icosahedron and dodecahedron the coordinates of vertices and the angle between these vectors involve Golden Mean $\phi = (1+\sqrt{5})/2$ so that algebraic extension must involve $\sqrt{5}$ at least.

   The dihedral angle θ between the faces of Platonic solid $\{p,q\}$ is given by $sin(\theta/2) = cos(\pi/q)/sin(\pi/p)$. For tetrahedron, cube and octahedron $sin(\theta)$ and $cos(\theta)$ involve $\sqrt{3}$. For icosahedron dihedral angle is $tan(\theta/2) = \phi$. For instance, the geometry of tetrahedron involves both $\sqrt{2}$ and $\sqrt{3}$. For dodecahedron more complex algebraic numbers are involved.

2. The rotation matrices for for the triangles of tetrahedron and icosahedron involve $cos(2\pi/3)$ and $sin(2\pi/3)$ associated with the quantum phase $q = exp(i2\pi/3)$ associated with it. The rotation matrices performing rotation for a pentagonal face of dodecahedron involves $cos(2\pi/5)$ and $sin(2\pi/5)$ and thus $q = exp(i2\pi/5)$ characterizing the extension. Both $q = exp(i2\pi/3)$ and $q = exp(i2\pi/5)$ are thus involved with icosahedral and dodecahedral rotation matrices. The rotation matrices for cube and for octahedron have rational matrix elements.

3. The Dynkin diagrams characterize both the finite discrete groups of $SU(2)$ and those of ADE groups. The Dynkin diagrams of Lie groups reflecting the structure of corresponding Weyl groups involve only the angles $\pi/2, 2\pi/3, \pi-\pi/6, 2\pi-\pi/6$ between the roots. They would naturally relate to quadratic extensions.

   For ADE Lie groups the diagram tells that the roots associated with the adjoint representation are either orthogonal or have mutual angle of $2\pi/3$ and have same length so that length ratios are equal to 1. One has $sin(2\pi/3) = \sqrt{3}/2$. This suggests that $\sqrt{3}$ belongs to the algebraic extension associated with ADE group always. For the non-simply laced Lie groups of type B, C, F, G the ratios of some root lengths can be $\sqrt{2}$ or $\sqrt{3}$.

For ADE groups assignable to n-polygons ($n > 5$) Galois group must involve the cyclic extension defined by $exp(i2\pi/n)$. The simplest option is that the extension corresponds to the roots of the polynomial $x^n = 1$.

3.5 A possible connection with number theoretic Langlands correspondence

I have discussed number theoretic version of Langlands correspondence in [K5, K22] trying to understand it using physical intuition provided by TGD (the only possible approach in my case). Concerning my unashamed intrusion to the territory of real mathematicians I have only one excuse: the number theoretic vision forces me to do this.

Number theoretic Langlands correspondence relates finite-dimensional representations of Galois groups and so called automorphic representations of reductive algebraic groups defined also for adeles, which are analogous to representations of Poincare group by fields. This is kind of relationship can exist follows from the fact that Galois group has natural action in algebraic reductive group defined by the extension in question.

The "Resiprocity conjecture" of Langlands states that so called Artin L-functions assignable to finite-dimensional representations of Galois group *Gal* are equal to L-functions arising from so called automorphic cuspidal representations of the algebraic reductive group G. One would have correspondence between finite number of representations of Galois group and finite number of cuspidal representations of G.

This is not far from what I am naively conjecturing on physical grounds: finite-D representations of Galois group are reductions of certain representations of G or of its subgroup defining the analog of spin for the automorphic forms in G (analogous to classical fields in Minkowski space). These representations could be seen as induced representations familiar for particle physicists dealing with Poincare invariance. McKay correspondence encourages the conjecture that the allowed spin representations are irreducible also with respect to *Gal*. For a childishly naive physicist knowing nothing about the complexities of the real mathematics this looks like an attractive starting point hypothesis.

In TGD framework Galois group could provide a geometric representation of "spin" (maybe even spin 1/2 property) as transformations permuting the sheets of the space-time surface identifiable as Galois covering. This geometrization of number theory in terms of cognitive representations analogous to the use of algebraic groups in Galois correspondence might provide a totally new geometric insights to Langlands correpondence. One could also think that Galois group represented in this manner could combine with the dynamical Kac-Moody group emerging from SSA to form its Langlands dual.

Skeptic physicist taking mathematics as high school arithmetics might argue that algebraic counterparts of reductive Lie groups are rather academic entities. In adelic physics the situation however changes completely. Evolution corresponds to a hierarchy of extensions of rationals reflected directly in the physics of dark matter in TGD sense: that is as phases of ordinary matter with $h_{eff}/h = n$ identifiable as divisor of the order of Galois group for an extension of rationals. Algebraic groups and their representations get physical meaning and also the huge generalization of their representation to adelic representations makes sense if TGD view about consciousness and cognition is accepted.

In attempts to understand what Langlands conjecture says one should understand first the rough meaning of many concepts. Consider first the Artin L-functions appearing at the number theoretic side. Consider first the Artin L-functions appearing at the number theoretic side.

1. L-functions (see `http://tinyurl.com/y8dc4zv9`) are meromorphic functions on complex plane that can be assigned to number fields and are analogs of Riemann zeta function factorizing into products of contributions labelled by primes of the number field. The definition of L-function involves Direchlet characters: character is very general invariant of group representation defined as trace of the representation matrix invariant under conjugation of argument.

2. In particular, there are Artin L-functions (see `http://tinyurl.com/y7thhodk`) assignable to the representations of *non-Abelian* Galois groups. One considers finite extension L/K of fields with Galois group G. The factors of Artin L-function are labelled by primes p of K. There are two cases: p is un-ramified or ramified depending on whether the number of primes of L to which p decomposes is maximal or not. The number of ramified primes is finite and in TGD framework they are excellent candidates for physical preferred p-adic primes for given extension of rationals.

   These factors labelled by p analogous to the factors of Riemann zeta are identified as characteristic polynomials for a representation matrix associated with any element in a preferred conjugacy class of G. This preferred conjugacy class is known as Frobenius element $Frob(p)$ for a given prime ideal p , whose action on given algebraic integer in O_L is represented as its p:th power. For un-ramified p the characteristic polynomial is explicitly given as determinant $det[I - t\rho(Frob(p))]^{-1}$, where one has $t = N(p)^{-s}$ and $N(p)$ is the field norm of p in the extension L (see `http://tinyurl.com/o4saw2l`).

   In the ramified case one must restrict the representation space to a sub-space invariant under inertia subgroup, which by definition leaves invariant integers of O_L/p that is the lowest part of integers in expansion of powers of p.

At the other side of the conjecture appear representations of algebraic counterparts of reductive Lie groups and their L-functions and the two number theoretic and automorphic L-functions would be identical.

1. Automorphic form F generalizes the notion of plane wave invariant under discrete subgroup of the group of translations and satisfying Laplace equation defining Casimir operator for translation group. Automorphic representations can be seen as analogs for the modes of classical fields with given mass having spin characterized by a representation of subgroup of Lie group G ($SO(3)$ in case of Poincare group).

   Automorphic functions as field modes are eigen modes of some Casimir operators assignable to G. Algebraic groups would in TGD framework relate to adeles defined by the hierarchy of extensions of rationals (also roots of e can be considered in extensions). Galois groups have natural action in algebraic groups.

2. Automorphic form (see `http://tinyurl.com/create.php`) is a complex vector valued function F from topological group to some vector space V. F is an eigen function of certain Casimir operators of G. In the simplest situation these function are invariant under a discrete subgroup $\Gamma \subset G$ identifiable as the analog of the subgroup defining spin in the case of induced representations.

   In general situation the automorphic form F transforms by a factor j of automorphy under Γ. The factor can also act in a finite-dimensional representation of group Γ, which would suggest that it reduces to a subgroup of Γ obtained by dividing with a normal subgroup. j satisfies 1-cocycle condition $j(g_1, g_2g_3) = j(g_1g_2, g_3)$ in group cohomology guaranteeing associativity (see `http://tinyurl.com/on7ffy9`). Cuspidality relates to the conditions on the growth of F at infinity.

3. Elliptic functions in complex plane characterized by two complex periods are meromorphic functions of this kind. A less trivial situation corresponds to non-compact group $G = SL(2, R)$ and $\Gamma \subset SL(2, Q)$.

There are more groups involved: Langlands group L_F and Langlands dual group ${}^LG$. A more technical formulation says that the automorphic representations of a reductive Lie group G correspond to homomorphisms from so called Langlands group L_F (see `http://tinyurl.com/ycnhkvm2`) at the number theoretic side to L-group ${}^LG$ or Langlands dual of algebraic G at group theory side (see `http://tinyurl.com/ycnk9ga5`). It is important to notice that ${}^LG$ is a complex Lie group. Note also that homomorphism is a representation of Langlands group L_F in L-group ${}^LG$. In TGD this would be analogous to a homomorphism of Galois group defining it as subgroup of the group G defining Kac-Moody algebra.

1. Langlands group L_F of number field is a speculative notion conjectured to be a extension of the Weil group of extension, which in turn is a modification of the absolute Galois group. Unfortunately, I was not able to really understand the Wikipedia definition of Weil group (`http://tinyurl.com/hk74sw7`). If E/F is finite extension as it is now, the Weil group would be $W_{E/F} = W_F/W_E^c$, W_E^c refers to the commutator subgroup W_E defining a normal subgroup, and the factor group is expected to be finite. This is not Galois group but should be closely related to it.

   Only finite-D representations of Langlands group are allowed, which suggests that the representations are always trivial for some normal subgroup of L_F For Archimedean local fields L_F is Weil group, non-Archimedean local fields L_F is the product of Weil group of L and of SU(2). The first guess is that SU(2) relates to quaternions. For global fields the existence of L_F is still conjectural.

2. I also failed to understand the formal Wikipedia definition of the L-group ${}^LG$ appearing at the group theory side. For a reductive Lie group one can construct its root datum $(X^*, \Delta, X_*, \Delta^c)$, where X^* is the lattice of characters of a maximal torus, X_* its dual, Δ the roots, and Δ^c the co-roots. Dual root datum is obtained by switching X^* and X_* and Δ and Δ^c. The root datum for G and ${}^LG$ are related by this switch.

   For a reductive G the Dynkin diagram of ${}^LG$ is obtained from that of G by exchanging the components of type B_n with components of type C_n. For simple groups one has $B_n \leftrightarrow C_n$. Note that for ADE groups the root data are same for G and its dual and it is the Kac-Moody counterparts of ADE groups, which appear in McKay correspondence. Could this mean that only these are allowed physically?

3. Consider now a reductive group over some field with a separable closure K (say k for rationals and K for algebraic numbers). Over K G as root datum with an action of Galois group of K/k. The full group ${}^LG$ is the semi-direct product ${}^LG^0 \rtimes Gal(K/k)$ of connected component as Galois group and Galois group. $Gal(K/k)$ is infinite (absolute group for rationals). This looks hopelessly complicated but it turns it that one can use the Galois group of a finite extension over which G is split. This is what gives the action of Galois group of extension (l/k) in ${}^LG$ having now finitely many components. The Galois group permutes the components. The action is easy to understand as automorphism on Gal elements of G.

Could TGD picture provide additional insights to Langlands duality or vice versa?

1. In TGD framework the action of Gal on algebraic group G is analogous to the action of Gal on cognitive representation at space-time level permuting the sheets of the Galois covering, whose number in the general case is the order of Gal identifiable as $h_{eff}/h = n$. The connected component ${}^LG0$ would correspond to one sheet of the covering.

2. What I do not understand is whether ${}^LG = G$ condition is actually forced by physical contraints for the dynamical Kac-Moody algebra and whether it relates to the notion of measurement resolution and inclusions of HFFs.

3. The electric-magnetic duality in gauge theories suggests that gauge group action of G on electric charges corresponds in the dual phase to the action of ${}^LG$ on magnetic charges. In self-dual situation one would have $G =^L G$. Intriguingly, CP_2 geometry is self-dual (Kähler form is self-dual so that electric and magnetic fluxes are identical) but induced K̈ahler form is self-dual only at the orbits of partonic 2-surfaces if weak form of electric-magnetic duality holds true. Does this condition leads to ${}^LG = G$ for dynamical gauge groups? Or is it possible to distinguish between the two dynamical descriptions so that Langlands duality would correspond to electric-magnetic duality. Could this duality correspond to the proposed duality of two variants of SH: namely, the electric description provided by string world sheets and magnetic description provided by partonic 2-surfaces carrying monopole fluxes?

3.6 A formulation of adelic TGD in terms of cognitive representations?

The vision about p-adic physics as cognitive representations of real physics [L4] encourages to consider an amazingly simple formulation of TGD diametrically opposite to but perhaps consistent with the vision based on the notion of WCW and WCW spinor fields. Finiteness of cognitive and measurement resolutions would not be enemies of the theoretician but could make possible to deduce highly non-trivial predictions from the theory by getting rid of all irrelevant information and using only the most significant bits. Number theoretic physics need not of course cover the entire quantum physics and could be analogous to topological quantum field theories: even this might provide huge amounts of precise information about the quantum physics of TGD Universe.

3.6.1 Could the discrete variant of WCW geometry make sense?

The first thing that one can imagine is number theoretic discretization of WCW by assuming that WCW coordinates belong to an extension of rationals. Integration would reduce to a summation but the problem is that there are too many points in the extension so that sums do not make sense in real sense. In the case of space-time surfaces the problems are solved by the fact that space-time surfaces have dimension lower than the imbedding space and the number of points with coordinates in the extension is in typical case finite: exceptions are surfaces such as canonically imbedded M^4 or CP_2. This option does not work at the level of WCW.

Cognitive representations however carry information about the points with coordinates in the extension of rationals defining the adele and possibly about the directions of strings emanating from these points. The effective WCW is kind of coset space with most of degrees of freedom not visible in the cognitive representation. Cognitive representations would specify the points in the extension of rationals for space-time surface, string world sheets, or even for their intersection with partonic surfaces at the ends of CD carrying fermion number plus those at the ends of sub-CDs forming a hierarchy.

Could one use the points of cognitive representation as coordinates for this effective WCW so that everything including WCW integration would reduce to well-defined summations? This would solve the problem of too many points in sub-WCW associated with the extension. Could one formulate everything that one can know at given level of cognitive hierarchy defined by extensions?

This idea was already suggested by the interpretation of p-adic mass calculations.

1. p-Adic mass calculations would correspond to cognitive representation of real physics [K1, K6]. For large p-adic primes p-adic thermodynamics converges extremely rapidly as powers $p^{-n/2}$ and the results from two lowest orders are practically exact.

2. What is however required is a justification for the map of p-adic mass squared values to real numbers by canonical identification. Quite generally this map makes sense for group invariants - say Lorentz invariants defined by inner products of momenta. As a matter of fact, the construction of quantum algebras and Yangians demands p-adic topology for the antipode to exist mathematically so that this approach could be forced by mathematical consistency [B1].

3.6.2 Could scattering amplitudes be constructed in terms of cognitive representations?

The crazy looking idea that cognitive representations defined by common points of real and p-adic variants of space-time surfaces or even partonic 2-surfaces is at least worth of showing to be wrong. If the idea works, cognitive representations could code what can be known about classical and even quantum dynamics and reduce physics to number theory. Also WCW would be discretized with points of discretized space-time surface defining WCW coordinates. Functional integral over WCW would reduce to a converging sum over cognitive representations.

It is interesting to look what this could mean if scattering amplitudes correspond in some sense to algebraic computations in bi-algebra besides product also co-product as its time reversal and interpreted as 3-vertex physically.

1. For the simplest option fermions would reside at the intersection points of partonic 2-surfaces and string world sheets. One possibility considered earlier is that at these points the Galois coverings are singular meaning that all sheets co-incide. This might be too strong condition and might be replacable by a weaker condition that Galois group at these points reduces to its sub-group and normal subgroup leaves amplitudes invariant. A reduction of measurement resolution would be in question.

2. If the basic computational operation involves a fusion of representations of Galois group, fusion algebra could describe the situation [L3]. The Galois groups assignable to the incoming lines of 3-vertex must correspond to Galois groups, which define groups of 3-levelled hierarchy of extension of rationals allowing inclusion homomorphism. Therefore the values of Planck constant would be of from $h_{eff}/h \in \{n_1, n_1n_2, n_1n_2n_3\}$. The tensor product decomposition would tell the outcome of tensor product. One can consider also 2-vertices corresponding to a phase transition $n_1 \leftrightarrow n_1n_2$ changing the value of h_{eff}/h.

   McKay graphs (see `http://tinyurl.com/z48d92t`) for Galois groups describe the decomposition of the tensor products of representations of Galois groups. In general the tensor products for corresponding KMAs restricted to Galois group are not irreducible. What could this mean? Are they allowed to occur? Are there general results allowing to conclude how do the analogs of McKay graphs for the tensor products of the irreps of the group defining Kac-Moody group relate to the McKay graphs for its finite discrete sub-groups?

Possible problems relate to the description of momenta and higher excitations of SKMAs. In topological QFTs one loses information about metric properties such as mass but what happens in number theoretic QFT? Could the Galois approach expanded to include also discrete variants of quaternions and octonions assigna ble to extensions of rationals allow also the number theoretic description of also momenta?

1. Octonions and quaternions have $G(2)$ and $SO(3)$ as automorphisms groups (analogs of Galois groups). The octonionic automorphisms respecting chosen imaginary consist of $SU(3)$ rotations. These groups would be replaced with their dicrete variants with matrix elements in an extension of rationals.

   The automorphism group Gal for the extension of rationals and automorphism group $Aut \in \{G_2, SU(3), SO(3)\}$ for octonions/for octonions with fixed unit/for quaternions form a semi-direct product $Gal \rtimes Aut$ with multiplication rule $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_2g_1(g_b))$, where $g_1(g_b)$ represents the element of Aut obtained by performing Gal automorphism g_1 for g_b. For rational elements g_b one has $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_ag_b)$ so that Gal Aut_Q commute. An interesting possibility is that the automorphisms of $Aut \in \{SU(3), SO(3)\}$ can be interpreted in terms of standard model symmetries whereas Gal would relate to the dynamical symmetries.

   In M^8 picture one has naturally wave functions in the space of quaternionic light-like 8-momenta and it is natural to decompose quaternionic momenta to longitudinal M^2 piece and transversal E^2 piece. The physical interpretation of this condition has been discussed thoroughly in [K26]. One has thus more than mere analog of TQFT.

2. If fermions propagate along the lines of the TGD analogs twistor graphs, one must have an analog of propagator. Twistor approach [K26] implies that the propagator is replaced with the inverse of the fermion propagator for quaternionic 8-momentum as a residue with sigma matrices representing the quaternionic units. This is non-vanishing only if the fermion chirality is "wrong". This has co-homological interpretation: for external lines the inverse of the propagator would annihilate the state (co-closedness) unlike for internal lines.

3. Triality holds true for the octonionic vector representation assignable to momenta and octonionic spinors and their conjugates. All these should be quaternionic, in other words belong to some complexified quaternionic $M^4 \subset M^8$. The components of these spinors should belong to an extension of rational used with imaginary unit commuting with octonionic imaginary units.

4. The condition that the amplitudes belong to an extension of rationals could be extremely powerful when combined with category theoretic view implying the Hilbert space isometries allowing to relate amplitudes at different levels of the hierarchy. This conditions should be true also for the twistors in terms which momenta can be expressed. Also the space $SU(3)/U(1) \times U(1)$ of CP_2 twistors would be replaced with a sub-space with points in an extension of rationals.

4 Could McKay correspondence generalize in TGD framework?

McKay correspondence is rather mysterious looking correspondence appearing in several fields. This correspondence is extremely interesting from point of view of adelic TGD [L5] [L4].

1. McKay graphs code for the fusion algebra of irreducible representations (irreps) of finite groups (see `http://tinyurl.com/z48d92t`). For finite subgroups of $G \subset SU(2)$ McKay graphs are extended Dynkin diagrams for affine (Kac-Moody) algebras of ADE type coding the structure of the root diagram for these algebras. The correspondence looks mysterious since Dynkin diagrams have quite different geometric interpretation.

2. McKay graphs for finite subgroups of $G \subset SU(2)$ characterize also the fusion rules of minimal conformal field theories (CFTs) having Kac-Moody algebra (KMA) of $SU(2)$ as symmetries (see `http://tinyurl.com/y7doftpe`). Fusion rules characterize the decomposition of the tensor products of primary fields in CFT. For minimal CFTs the primary fields belonging to the irreps of $SU(2)$ are in 1-1 correspondence with irreps of G, and the fusion rules for primary fields are same as for the irreps of G. The irreps of $SU(2)$ are also irreps of G.

   Could the ADE type affine algebra appear as dynamical symmetry algebra too? Could the primary fields for ADE defining extended ADE Cartan algebra be constructed as G-invariants formed from the irreps of G and be exponentiated using the standard free field construction using the roots of the ADE KMA a give ADE KMA acting as dynamical symmetries?

3. McKay graphs for $G \subset SU(2)$ characterize also the double point singularities of algebraic surfaces of real dimension 4 in C^3 (or CP^3 , one variant of twistor space!) with real dimension 6 (see `http://tinyurl.com/ydz93hle`). The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This partially explains the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of spheres is the number of irreps for G is not obvious to me.

   The double point singularities of $C^2 \subset C^3$ allow thus ADE classification. The number of added points corresponds to the dimension of Cartan algebra for ADE type affine algebra, whose Dynkin diagram codes for the finite subgroup $G \subset SU(2)$ leaving the algebraic surface looking locally like C^2 invariant and acting as isotropy group of the singularity.

These results are highly inspiring concerning adelic TGD.

1. The appearance of Dynkin diagrams in the classification of minimal CFTs inspires the conjecture that in adelic physics Galois groups Gal or semi-direct products $G \triangleleft Gal$ of Gal with a discrete subgroup G of automorphism group $SO(3)$ (having $SU(2)$ as double covering!) classifies TGD generalizations of minimal CFTs. Also discrete subgroups of octonionic automorphism group can be considered. The fusion algebra of irreps of Gal would define also the fusion algebra for KMA for the counterparts of minimal fields. This would provide deep insights to the general structure of adelic physics.

2. One cannot avoid the question whether the extended ADE diagram could code for a dynamical symmetry of a minimal CFT or its modification? If the Gal singlets formed from the primary fields of minimal model define primary fields in Cartan algebra of ADE type KMA, then standard free field construction would give the charged KMA generators. In TGD framework this conjecture generalizes.

3. A further conjecture is that the singularities of space-time surface imbedded as 4-surface in its 6-D twistor bundle with twistor sphere as fiber could be classified by McKay graph of Gal. The singular intersection of the Euclidian and Minkowskian regions of space-time surface is especially interesting: the twistor spheres at the common points defining light-like partonic orbits need not be same but have intersections with intersection matrix given by McKay graph for Gal. The basic information about adelic CFT would be coded by the general character of singularities for the twistor bundle.

4. In TGD also singularities in which the group *Gal* is reduced to its subgroup Gal/H, where H is normal group are possible and would correspond to phase transition reducing the value of Planck constant. What happens in these phase transitions to single particle states would be dictated by the decomposition of representations of *Gal* to those of Gal/H and transition matrix elements could be evaluated.

One can find from web excellent articles about the topics to be discussed in this article.

1. The article "*Cartan matrices, finite groups of quaternions, and Kleinian singularities*" of John McKay [A3] (see `http://tinyurl.com/ydygjgge`) summarizes McKay correspondence.

2. Miles Reid has written an article " *The Du Val singularities* A_n, D_n, E_6, E_7, E_8" [A4] (see `http://tinyurl.com/ydz93hle`). Also the article " *Chapters on algebraic surfaces*" [A5](see `http://tinyurl.com/yaty9rzy`) of Reid should be helpful. There is also an article "*Resolution of Singularities in Algebraic Varieties*" [A2] (see `http://tinyurl.com/yb7cuwkf`) of Emma Whitten about resolution of singularities.

3. Andrea Cappelli and Jean-Benard Zuber have written an article "*A-D-E Classification of Conformal Field Theories*" [B2] about ADE classification of minimal CFT models (see `http://tinyurl.com/y7doftpe`).

4. McKay correspondence appears also in M-theory, and the thesis "*On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective*" [B3] (see `http://tinyurl.com/ycmyjukn`) of Yang-Hui He might be helful for the reader. In this work the possible generalization of McKay correspondence so that it would apply form finite subgroups of $SU(n)$ is discussed. $SU(3)$ acting as subgroup of automorphism group G_2 of octonions is especially interesting in this respect. The idea is rather obvious: the fusion diagram for the theory in question would be the McKay graph for the finite group in question.

4.1 McKay graphs in mathematics and physics

McKay graphs for subgroups of $SU(2)$ reducing to Dynkin diagrams for affine Lie algebras of ADE type appear in several manners in mathematics and physics.

4.1.1 McKay graphs

McKay graphs [A3] (see `http://tinyurl.com/ydygjgge`) code for the fusion algebra of irrpes of finite groups G (for Wikipedia article see `http://tinyurl.com/z48d92t`). One considers the tensor products of irreps with the canonical representation (doublet representation for the finite sub-groups of $SU(2)$), call it V. The irreps V_i correspond to nodes and their number is equal to the number of irreps G.

Two nodes i and j are no connected if the decomposition of $V \otimes V_i$ to irreps does not contain V_j. There is arrow pointing from $i \rightarrow j$ in this case. The number $n_{ij} > 0$ or number of arrows tells how many times j is contained in $V \otimes V_j$. For $n_{ij} = n_{ji}$ there is no arrow.

One can characterize the fusion rules by matrix $A = d\delta_{ij} - n_{ij}$, where d is the dimension of the canonical representation. The eigenvalues of this matrix turn out to be given by $d - \xi_V(g)$, where $\xi_V(g)$ is the character of the canonical representation, which depends on the conjugacy class of g only. The number of eigenvalues is therefore equal to the number $n(class, G)$ of conjugacy classes. The components of eigenvectors in turn are given by the values $\chi_i(g)$ of characters of irreps.

4.1.2 MacKay graphs and Dynkin diagrams

The nodes of the Dynkin diagram (see `http://tinyurl.com/hpm5y9s`) are positive simple root vectors identified as vectors formed by the eigenvalues of the Cartan sub-algebra generators under adjoint action on Lie algebra. In the case of affine Lie algebra the Cartan algebra contains besides the Cartan algebra of the Lie group also scaling generator $L_0 = td/dt$ and the number of nodes increases by one.

The number of positive simple roots equals to the dimension of the root space. The number n_{ij} codes now for the angle between positive simple roots. The number of edges connecting root vectors is $n = 0, 1, 2, 3$ depending on whether the angle between root vectors is $\pi/2, 2\pi/3, 3\pi/4$, or $5\pi/6$. The ratios of lengths of connected roots can have values $\sqrt{n}$, $n \in \{1, 2, 3\}$, and the number n of edges corresponds to this ratio. The arrow is directed to the shorter root if present. For simply laced Lie groups (ADE groups) the roots have unit length so that only single undirected edge can connect the roots. Weyl group acts as symmetries of the root diagram as reflections in hyperplanes orthogonal to the roots.

The Dynkin diagrams of affine algebras are obtained by adding to the Cartan algebra a generator which corresponds to the scaling generator $L_0 = td/dt$ of affine algebra assumed to act via adjoint action to the Lie algebra. Depending on the position of the added node one obtains also twisted versions of the KMA.

For the finite subgroups of $SU(2)$ the McKay graphs reduce to Dynkin diagrams of affine Lie algebras of ADE type [A3] (see `http://tinyurl.com/ydygjgge`) so that one has either $n_{ij} = 0$ or $n_{ij} = 1$ for $i \neq j$. There are no self-loops ($n_{ii} \neq 0$). The result looks mysterious since the two diagrams describe quite different things. One can also raise the question whether ADE type affine algebra might somehow emerge in minimal CFT involving $SU(2)$ KMA for which ADE classification emerges.

In TGD framework the interpretation of finite groups $G \subset SU(2)$ in terms of quaternions is an attractive possibility since rotation group $SO(3)$ acts as automorphisms of quaternions and has $SU(2)$ as its covering group.

4.1.3 ADE diagrams and subfactors

ADE classification emerges also naturally for the inclusions of hyper-finite factors of type II_1 [K14, K3]. Subfactors with index smaller than four have so called principal graphs characterizing the sequence of inclusions equal to one of the A, D or E Coxeter-Dynkin diagrams: see the article "*In and around the origin of quantum groups*" of Vaughan Jones [A6] (see `http://tinyurl.com/ycbbbvpq`). As a matter of fact, only the D_{2n} and E_6 and E_8 do occur. It is also possible to construct $M : N = 4$ sub-factor such that the principle graph is that for any subgroup $G \subset SU(2)$. This suggests that the subfactors $M : N = 4cos^2(\pi/n) < 4$ correspond to quantum groups. The basic objects can be seen as quantum spinors so that again the appearance of subgroups of $SU(2)$ looks natural. One can still wonder whether ADE KMAs might be involved.

4.1.4 ADE classification for minimal CFTs

.

CFTs on torus [B2] are characterized by modular invariant partition functions, which can be expressed in terms of characters of the scaling generator L_0 of Virasoro algebra (VA) given by

$$Z(\tau) = Tr(X) \ , \quad X = exp\{i2\pi \left[\tau(L_0 - c/24) - \overline{\tau}(\overline{L}_0 - c/24)\right]\} \ . \tag{4.1}$$

Modular invariance requires that $Z(\tau)$ is invariant under modular transformations leaving the conformal equivalence class of torus invariant. Modular group equals to $SL(2, Z)$ has as generators the transformations $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -1/\tau$. The partition function can be expressed as

$$Z(\tau) = \sum N_{j\overline{j}} \chi_j(q) \chi_{\overline{j}}(\overline{q}) \ , \quad q = exp(i2\pi\tau) \ , \quad \overline{q} = exp(-i2\pi\overline{\tau}) \ . \tag{4.2}$$

Here χ_j corresponds to the trace of $L_0 - c/24$ for a representation of KMA inducing the VA representation. Modular invariance of the partition function requires $SNS^\dagger = N$ and $TNT^\dagger = N$.

The ADE classification for minimal conformal models summarized in [B2] (see `http://tinyurl.com/y7doftpe`) involves $SU(2)$ affine algebra with central extension parameter k. The central extension parameter for the VA is $c < 1$. The fusion algebra for primary fields in representations of $SU(2)$ KMA characterizes the CFT to a high degree.

The fusion rules characterized the decomposition of the tensor product of representation D_i with representation D_j as $i \otimes j = N_{ij}^k D_k$. Due to the properties of the tensor product the matrices $\mathcal{N}_i = N_{ij}^k$ form and associative and commutative algebra and one can diagonalize these matrices simultaneously. This algebra is known as Verlinde algebra and its elements can be expressed in terms of unitary modular matrix S_{ij} representing the transformation of characters in the modular transformation $\tau \to -1/\tau$.

The generator of the Verlinde algebra is fusion algebra for the 2-D representation of $SU(2)$ generating the fusion algebra (this corresponds to the fact that tensor powers of this representations give rise to all representations of $SU(2)$). It turns out that for minimal models with a finite number of primary fields (KMA representations) the fusion algebra of KMA reduces to that for a finite subgroup of $SU(2)$ and thus corresponds to ADE KMA. The natural interpretation is that the condition that the number of primary fields is finite is realized if the primary fields correspond also to the irreps of finite subgroup of SU(2).

Could the ADE type KMA actually correspond to a genuine dynamical symmetry of minimal CFT? For this conjecture makes sense, the roots of ADE type KMA should be in 1-1 correspondence with the irreps of $G \subset SU(2)$ assignable to primary fields. How could this be possible? In the free field construction of ADE type KMA generators one constructs charged KMA generators from free fields in Cartan algebra by exponentiating the quantities $\alpha \cdot \phi$, where α is the root and ϕ is a primary field corresponding to the element of Cartan algebra of KMA. Could $SU(2)$ invariants formed from the primary fields defined by each G- (equivalently $SU(2)$-) multiplet give rise to $SU(2)$ neutral multiplet of primary fields of ADE type Cartan algebra and could their exponentiation give rise to ADE type KMA acting as dynamical symmetries of a minimal CFT?

4.1.5 The resolution of singularities of algebraic surfaces and extended Dynkin diagrams of ADE type

The classification of singularities of algebraic surfaces leads also to extended Dynkin diagrams of ADE type.

1. Classification of singularities

In algebraic geometry the classification of singularities of algebraic varieties [A2] is a central task. The singularities of curves in plane represent simplest singularities (see `http://tinyurl.com/y8ub2c4s`). The resolution of singularities of complex curves in C^3 is less trivial task.

The resolution of singularity (`http://tinyurl.com/y8veht3p`) is a central concept and means elimination of singularity by modifying it locally. There is extremely general theorem by Hiroka stating that the resolution of singularities of algebraic varieties is always possible for fields with characteristic zero (reals and p-adic number fields included) using a sequence of birational transformations. For finite groups the situation is unclear for dimensions $d > 3$.

The articles of Reid [A4] and Whitten [A2] describe the resolution for algebraic surfaces (2-D surfaces with real dimension equal to four). The article of Reid describes how the resolutions of double-point singularities of $m = d_c = 2$-D surfaces in $n = d_c = 3$-D C^3 or CP_3 (d_c refers to complex dimension) are classified by ADE type extended Dynkin diagrams. Subgroups $G \subset SU(2)$ appear naturally because the surface has dimension $d_c = 2$. This is the simplest non-trivial situation since for Riemann surface with $(m, n) = (1, 2)$ the group would be discrete subgroup of $U(1)$.

2. Singularity and Jacobians

What does one mean with singularity and its resolution? Reid [A4] (see `http://tinyurl.com/ydz93hle`) discusses several examples. The first example is the singularity of the surface $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$.

1. One can look the situation from the point of view of imbedding of the 2-surface to C^3: one considers map from tangent space of the surface to the imbedding space C^3. The Jacobian of the imbedding map $(x_2, x_3) \to (x_1, x_2, x_3) = \pm\sqrt{x_2x_3}, x_2, x_3)$ becomes ill-defined at origin since the partial derivatives $\partial x_1/\partial x_2 = (\sqrt{x_3/x_2})/2$ and $\partial x_1/\partial x_3 = (\sqrt{x_2/x_3})/2$ have all possible limiting values at singularity. The resolution of singularity must as a coordinate transformation singular at the origin should make the Jacobian well-defined. Obviously this must mean addition of points corresponding to the directions of various lines of the surface through origin.

2. A more elegant dual approach replaces parametric representation with representation in terms of conditions requiring function to be constant on the surface. Now the Jacobian of

a map from C^3 to the 1-D normal space of the singularity having polynomial $P(x_1, x_2, x_3)$ as coordinate is considered. Singularity corresponds to the situation when the rank of the Jacobian defined by partial derivatives is less than maximal so that one has $\partial P/\partial x_i = 0$. The resolution of singularity means that the rank becomes maximal. Quite generally, for co-dimension m algebraic surface the vanishing of polynomials P_i, $i = 1, ..., m$ defines the surface. At the singularity the reduction of the rank for the matrix $\partial P_i/\partial x_n$ from its maximal value takes place.

3. Blowing up of singularity

Codimension one algebraic surface is defined by the condition $P(x_1, x_2, ..., x_n) = 0$, where $P(x_1, ..., x_n)$ is polynomial. For higher codimensions one needs more polynomials and the situation is not so neat anymore since so called complete intersection property need not hold anymore. Reid [A4] gives an easy-to- understand introduction to the blowing up of double-point singularities. Also the article "*Resolution of Singularities in Algebraic Varieties"* of Emma Whitten [A2] (see `http://tinyurl.com/yb7cuwkf`) is very helpful.

1. Coordinates are chosen such that the singularity is at the origin $(x, y, z) = (0, 0, 0)$ of complex coordinates. The polynomial has vanishing linear terms at singularity and the first non-vanishing term is second power of some coordinate, say x_1, so that one has $x_1 = \pm\sqrt{P_1(x_1, x_2, x_3}$, where x_1 in P_1 appears in powers higher than 2. At the singularity the two roots co-incide. One can of course have also more complex singularities such as triple-points.

2. The simplest example $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$ has been already mentioned. This singularity has the structure of double cone since one as $x_1 = \pm\sqrt{x_2x_3}$. At $(0, 0, 0)$ the vertices of the two cones meet.

3. One can look this particular situation from the perspective of projective geometry. Homogenous polynomials define a surface invariant under scalings of coordinates so that modulo scalings the surface can be regarded also as complex curve in CP_2. The conical surface can be indeed seen as a union of lines $(x_1 = k^2x_3, x_2 = kx_3)$, where k is complex number. The ratio $x_1 : x_2 : x_3$ for the coordinates at given line is determined by $x_1 : x_2 = k$ and $x_2 : x_3 = k$ so that the surface can be parameterized by k and the coordinate along given line.

   In this perspective the singularity decomposes to the directions of the lines going through it and the situation becomes non-singular. The replacement of the original view with this gives a geometric view idea about the resolution of singularity: the 2-surface is replaced by a bundle lines of surfaces going through the singularity and singularity is replaced with a union of directions for these lines.

Quite generally, in the resolution of singularity, origin is replaced by a set of points (x_1, x_2, x_3) with a well-defined ratio $(x_1 : x_2 : x_3)$. This interpretation applies also to more general singularities. One can say that origin is replaced with a projective sub-manifold of 2-D projective space CP_2 (very familiar to me)! This procedure is known as blowing up. Strictly speaking, one only replaces origin with the directions of lines in C^3.

Remark: In TGD the wormhole contacts connecting space-time sheets of many-sheeted space-time could be seen as outcomes of blowing up procedure.

Blowing up replaces the singular point with projective space CP_1 for which points with same value of $(x_1 : x_2 : x_3)$ are identified. Blowing up can be also seen as a process analogous to seeing the singularity such as self-intersection of curve as an illusion: the curve is actually a projection of a curve in higher dimensional space to which it is lifted so that the intersection disappears [A2] (see `http://tinyurl.com/yb7cuwkf`). Physicist can of course protest by saying that in space-time physics is is not allowed to introduce additional dimensions in this manner!

There is an analytic description for what happens at the singular point in blowing up process [A2] (see `http://tinyurl.com/yb7cuwkf`).

1. In blowing up one lifts the surface in higher-dimensional space $C^3 \times CP_2$ (C^3 can be replaced by any affine space). The blowing up of the singularity would be the set of lines $\overline{q}$ of the surface S going through the singularity that is the set $B = \{(q, \overline{q}) | q \in S\}$. This set can be seen

as a subset of $C^3 \times CP_2$ and one can represent it explicitly by using projective coordinates (y_1, y_2, y_3) for CP_2. Consider points of C^3 and CP_2 with coordinates $z = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The coordinate vectors must be parallel x is to be at line y. This requires that all 2×2 sub-determinants of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{4.3}$$

vanish: that is $x_i y_j - x_j y_i = 0$ for all pairs $i < j$. This description generalizes to the higher-dimensional case. The added CP_1s defined what is called exceptional divisor in the blown up surface. Recall that divisors (see `http://tinyurl.com/yc7x3ohx`) are by definition formal combinations of points of algebraic surface with integer coefficients. The principal divisors defined by functions are sums over their zeros and poles with integer weight equal to the order of zero (negative for pole).

The above example considers a surface $x_1^2 - x_2 x_3 = 0$ which allows interpretation as a projective surface. The method however works also for more general case since the idea about replacing point with directions is applied only at origin.

2. One can consider a more practical resolution of singularity by performing a bi-rational coordinate transformation becoming singular at the singular point. This can improve the singularity by blowing it up or make it worse by inducing blowing down. The idea is to perform a sequence of this kind of coordinate changes inducing blowing ups so that final outcome is free of singularities.

   Since one considers polynomial equations both blowing up and its reversal must map polynomials to polynomials. Hence a bi-rational transformation b acting as a surjection from the modified surface to the original one must be in question (for bi-rational geometry see `http://tinyurl.com/yadoo3ot`). At the singularity b is many-to-one y so that at this point inverse image is multivalued and gives rise to the blowing up.

   The equation $P(x_1, x_2, x_3) = 0$ combined with the equations $x_i y_j - x_j y_i = 0$ by putting $y_3 = 1$ (the coordinates are projective) leads to a parametric representation of S using y_1 and y_2 as coordinates instead of x_1 and x_2. Origin is replaced with CP_1. This representation is actually much more general. Whitten [A2] gives a systematic description of resolution of singularities using this representation. For instance, cusp singularity $P(x_1, x_2) = x_1^2 - x_2^3 = 0$ is discussed as a special case.

3. Topologically the blow up process corresponds to the gluing of CP_2 to the algebraic surface $A : A \rightarrow A\#CP_2$ and clearly makes it more complex. One can say that gluing occurs along sphere CP_1 and since the process involves several steps several spheres are involved with the resolution of singularities.

4. ADE classification for resolutions of double point singularities of algebraic surfaces

ADE classification emerges for co-dimension one double point singularities of complex surfaces in C^3 known as Du Val singularities. The surface itself can be seen locally as C^2. These surfaces are 4-D in real sense can have self-intersections with real dimension 2. In the singular point the dimension of the intersection is reduced and the dimension of tangent space is reduced (the rank of Jacobian is not maximal). The vertices of cone and cusp are good examples of singularities.

The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This helps to understand the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of projective lines equals to the number of irrepss of G appearing as nodes in McKay graph looks to me rather mysterious. Reid's article [A4] gives the characterization of groups G and canonical forms of the polynomials defining the singular surfaces.

The reason why Du Val singularities are so interesting from TGD point of view is that complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6.

The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces as preferred extremals have real dimension 4 and assumed possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure [K13].

4.2 Do McKay graphs of Galois groups give overall view about classical and quantum dynamics of quantum TGD?

McKay graphs for Galois groups are interesting from TGD view point for several reasons. Galois groups are conjectured to be the number theoretical symmetries for the hierarchy of extensions of rationals defining hierarchy of adelic physics [L5] [L4] and the notion of CFT is expected to generalize in TGD framework so that ADE classification for minimal CFTs might generalize to a classification of minimal number theoretic CFTS by Galois groups.

1. Vision

The arguments leading to the vision are roughly following.

1. Adelic physics postulates a hierarchy of quantum physics with adeles at given level associated with extension of rationals characterized partially by Galois group and ramified primes of extension. The dimension of the extensions dividing the order of Galois group is excellent candidate for defining the value of Planck constant $h_{eff}/h = n$ and ramified primes could correspond to preferred p-adic primes. The discrete sets of points of space-time surface for which imbedding space coordinates are in the extension define what I have interpreted as cognitive representations and can be said to be in the intersection of all number fields involved forming kind of book like structure with pages intersecting at the points with coordinates in extension.

   Galois groups would define a hierarchy of theories and the natural first guess is that Galois groups take the role of subgroups of $SU(2)$ in CFTs with $SU(2)$ KMA as symmetry. Could the MacKay graphs defining the fusion algebra of Galois group define the fusion algebra of corresponding minimal number theoretic QFTs in analogy with minimal conformal models? This would fix the primary fields of theories assignable to given level of adele hierarchy to be minimal representations of *Gal* perhaps having also interpretation as representations of KMAs or their generalization to TGD framework.

2. The analogies between TGD and the theory of Du Val singularities is intriguing. Complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces have real dimension 4 and possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure.

   The twistor bundle of space-time surface has 2-sphere CP_1 as a fiber and space-time surface as base [K21, K26]. Space-time surfaces can be realized as sections in their own 6-D twistor bundle obtained by inducing twistor structure from the product $T(M^4) \times T(CP_2)$ of twistor bundles of M^4 and CP_2. Section is fixed only modulo gauge choice, which could correspond to the choice of the Kähler form defining twistor structure from quaternionic units represented as points of S^2. Even if this choice is made, $U(1)$ gauge transformations remain and could correspond to gauge transformations of WCW changing its Kähler gauge potential by gradient and adding to Kähler function a real part of holomorphic function of WCW coordinates.

   If the imbedding of 4-D space-time surface as section can become singular in given gauge, it will have self-intersections with dimension 2 possibly assignable to partonic 2-surfaces and maybe also string world sheets playing a key role in strong form of holography (SH). Could SH mean that information about classical and quantum theory is coded by singularities of the imbedding of space-time surface to twistor bundle. This would be highly analogous to what happens in the case of complex functions and also in twistor Grassmann theory whether the amplitudes are determined by the data at singularities.

3. Where would the intersections take place? Space-time regions with Minkowskian and Euclidian signature of metric have light-like orbits of partonic 2-surfaces as intersections. These

surfaces are singular in the sense that the metric determinant vanishes and tangent space of space-time surface becomes effectively 3-D: this would correspond to the reduction of tangent space dimension of algebraic surface at singularity. It is attractive to think that the lifts of Minkowskian and Euclidian space-time sheets have twistor spheres, which only intersect and have intersection matrix represented by McKay graph of Gal.

What about string world sheets? Does it make sense to regard them as intersections of 4-D surfaces? This does not look plausible idea but there are also other characterizations of string world sheets. One can also ask about the interpretation of the boundaries of string world sheets, in particular the points at the partonic 2-surfaces. How could they relate to singularities? The points of cognitive representation at partonic 2-surfaces carrying fermion number should belong to cognitive representation with imbedding space coordinates belonging to an extension of rationals.

4. In Du Val theory the resolution of singularity means that one adds additional points to a double singularity: the added points form projective sphere CP_1. The blowing up process is like lifting self-intersecting curve to a non-singular curve by imbedding it into 3-D space so that the original curve is its projection. Could singularity disappear as one looks at 6-D objects instead of 4-D object? Could the blowing up correspond in TGD to a transition to a new gauge in which the self intersection disappears or is shifted on new place? The intersections of 4-surfaces in 6-space analogous to roots of polynomial are topologically stable suggesting that they can be only shifted by a new choice of gauge.

   Self-intersection be a genuine singularity if the spheres CP_1 defining the fibers of the twistor bundles of branches of the space-time surface do not co-incide in the 2-D intersection. In the generic case they would only intersect in the intersection. Could the McKay diagram of Galois group characterize the intersection matrix?

5. The big vision could be following. Galois groups characterize the singularities at given level of the adelic hierarchy and code for the multiplets of primary fields and for the analogs of their fusion rules for TGD counterparts of minimal CFTs. Note that singularities themselves identified as partonic 2-surfaces and possibly also light partonic orbits and possibly even string world sheets are not restricted in any manner.

This idea need not be so far-fetched as it might look at first.

1. One considers twistor lift and self-intersections indeed occur also in twistor theory. When the M^4 projections of two spheres of twistor space CP_3 (to which the geometric twistor space $T(M^4) = M^4 \times S^2$ has a projection) have light-like separation, they intersect. In twistor diagrams the intersection corresponds to an emission of massless particle.
2. The physical expectation is that this kind of intersections could occur also for the twistor bundle associated with the space-time surface. Most naturally, they could occur along the light-like boundary of causal diamond (CD) for points with light-like separation. They could also occur along the partonic orbits which are light-like 3-surfaces defining the boundaries between Minkowskian and Euclidian space-time regions. The twistor spheres at the ends of light-like curve could intersect.

Why the number of intersecting twistor spheres should reduce to the number $n(irred, Gal)$ of irreducible representations (irreps) of Gal, which equals to $n(Gal)$ in Abelian case but is otherwise smaller? This question could be seen as a serious objection.

1. Does it make sense to think that although there are $n(Gal)$ in the local fiber of twistor bundle, the part of Galois fiber associated with the twistor fiber CP_1 has only $n(irrep, Gal)$ CP_1:s and even that the spheres could correspond to irreps of Gal. I cannot invent any obvious objection against this. What would happen that Could this mean realization of quantum classical correspondence at space-time level.
2. There are $n(irrep, G)$ irreps and $\sum_i n_i^2 = n(G)$. n_i^2 points at corresponding sheet labelled by irrep. The number of twistor spheres collapsing to single one would be n_i for n_i-D irrep so

that instead of states of representations the twistor spheres would correspond to irrep. One would have analogy with the fractionization of quantum numbers. The points assignable to n_i-D representations would become effectively $1/n_i$-fractionized. At the level of base space this would not happen.

4.2.1 Phase transitions reducing h_{eff}/h

In TGD framework one can imagine also other kinds of singularities. The reduction of Gal to its subgroup Gal/H, where H is normal subgroup defining Galois group for the Gal as extension of Gal/H is one such singularity meaning that the H orbits of space-time sheets become trivial.

1. The action of Gal could reduce locally to a normal subgroup H so that Gal would be replaced with Gal/H. In TGD framework this would correspond to a phase transition reducing the value of Planck constant $h_{eff}/h = n(Gal)$ labelling dark matter phases to $h_{eff}/h = n(Gal/H) = n(Gal)/n(H)$. The reduction to Gal/H would occur automatically for the points of cognitive representation belonging to a lower dimensional extension having Gal/H as Galois group. The singularity would occur for the cognitive points of both space-time surface and twistor sphere and would be analogous to $n(H)$-point singularity.

2. A singularity of the discrete bundle defined by Galois group would be in question and is assumed to induce similar singularity of $n(Gal)$ -sheeted space-time surface and its twistor lift. Although the singularity would occur for the ends of strings it would induce reduction of the extension of rationals to Gal/H, which should also mean that string world sheets have representation with WCW coordinates in smaller extension of rationals.

3. This would be visible as a reduction in the spectrum of primary fields of number theoretic variant of minimal model. I have considered the possibility that the points at partonic 2-surfaces carrying fermions located at the ends of string world sheets could correspond to singularities of this kind. Could string world sheets could correspond to this kind of bundle singularities? This singularity would not have anything to do with the above described self-interactions of the twistor spheres associated with the Minkowskian and Euclidian regions meeting at light-like orbits of partonic 2-surfaces.

4. This provides a systematic procedure for constructing amplitudes for the phase transitions reducing $h_{eff}/h = n(Gal)$ to $h_{eff}/h = n(Gal/H)$. The representations of Gal would be simply decomposed to the representations of $Gal(G/H)$ in the vertex describing the phase transition. In the simplest 2-particle vertex the representation of Gal remains irreducible as representation of Gal/H. Transition amplitudes are given by overlap integrals of representation functions of group algebra representations of Gal restricted to Gal/H with those of Gal/H.

   The description of transitions in which particles with different Galois groups arrive in same diagram would look like follows. The Galois groups must form an increasing sequence $... \subset Gal_i = Gal_{i+1}/H_{i+1} \subset ....$ The representations of the largest Galois group would be decomposed to the representations of smallest Galois group so that the scattering amplitudes could be constructed using the fusion algebra of the smallest Galois group. The decomposition to should be associative and commutative and could be carried in many manners giving the same outcome at the final step.

4.2.2 Also quaternionic and octonionic automorphisms might be important

What about the role of subgroups of $SU(2)$? What roles they could have? Could also they classify singularities in TGD framework?

1. SU(2) is indeed realize as multiplication of quaternions. $M^8 - H$ correspondence suggests that space-time surfaces in M^8 can be regarded as associative or co-associative (normal space-is associative. Associative translates to quaternionic. Associativity makes sense also at the level of H although it is not necessary. This would mean that the tangent space of space-time surface has quaternionic structure and the multiplication by quaternions is makes sense.

2. The Galois group of quaternions is $SO(3)$ and has discrete subgroups having discrete subgroups of $SU(2)$ as covering groups. Quaternions have action on the spinors from which twistors are formed as pairs of spinors. Could quaternionic automorphisms be lifted to a an $SU(2)$ action on these spinors by quaternion multiplication? Could one imagine that the representations formed as tensor powers of these representations give finite irreps of discrete subgroups of $SU(2)$ defining ground states of $SU(2)$ KMA a representations and define the primary fields of minimal models in this manner?

3. Galois groups for extensions of rationals have automorphic action on $SO(3)$ and its algebraic subgroups replacing matrix elements with their automorphs: for subgroups represented by rational matrices the action is trivial. One would have analogs of representations of Lorentz group $SL(2,C)$ induced from spin representations of finite subgroups $G \subset SU(2)$ by Lorentz transformations realizing the representation in Lobatchevski space. Lorentz group would be replaced by Gal and the Lobatchevski spaces as orbit with the representation of Gal in its group algebra. An interesting question is whether the hierarchy of discrete subgroups of $SU(2)$ in McKay correspondence relates to quaternionicity.

G_2 acts as octonionic automorphisms and SU(3) appears as its subgroup leaving on octonionic imaginary unit invariant. Could these semi-direct products of Gal with these automorphism groups have some role in adelic physics?

4.2.3 About TGD variant of ADE classification for minimal models

I already considered the ADE classification of minimal models. The first question is whether the finite subgroups $G \subset SU(2)$ are replaced in TGD context with Galois groups or with their semi-direct products $G \triangleleft Gal$. Second question concerns the interpretation of the Dynkin diagram of affine ADE type Lie algebra. Does it correspond to a real dynamical symmetries.

1. Could the MacKay correspondence and ADE classification generalize? Could fusion algebras of minimal models for KMA associated with general compact Lie group G be classified by the fusion algebras of the finite subgroups of G. This generalization seems to be discussed in [B3] (see `http://tinyurl.com/ycmyjukn`).

2. Could the fusion algebra of Galois groupGal give rise to a generalization of the minimal model associated with a KMA of Lie group $G \supset Gal$. The fusion algebra of Gal would be identical with that for the primary fields of KMA for G. Galois groups could be also grouped to classes consisting of Galois groups appearing as a subgroup of a given Lie group G.

3. In TGD one has a fractal hierarchy of isomorphic supersymplectic algebras (SSAs) (the conformal weights of sub-algebra are integer multiples of those of algebra) with gauge conditions stating that given sub-algebra of SSA and its commutator with the entire algebra annihilates the physical states. The remnant of the full SSA symmetry algebra would be naturally KMA.

   The pair formed by full SSA and sub-SSA would correspond to pair formed by group G and normal subgroup H and the dynamical KMA would correspond to the factor group G/H. This conjecture generalizes: one can replace G with Galois group and $SU(2)$ KMA with a KMA continuing Gal as subgroup. One the other hand, one has also hierarchies of extensions of rationals such that $i+1$:th extension of rationals is extension of i:th extension. G_i is a normal subgroup of G_{i+1} so that the group $Gal_{i+1,i} = Gal_{i+1}/Gal_i$ acts as the relative Galois group for $i+1$:th extension as extensions of i:th extension.

   This suggest the conjecture that the Galois groups Gal_i for extension hierarchies correspond to the inclusion hierarchies $SSA_i \supset SSA_{i+1}$ of fractal sub-algebras of SSA such that the gauge conditions for SSA_i define a hierarchy KMA_i of dynamical KMAs acting as dynamical symmetries of the theory. The fusion algebra of KMA_i theory would be characterized by Galois group Gal_i.

4. I have considered the possibility that the McKay graphs for finite subgroups $G \subset SU(2)$ indeed code for root diagrams of ADE type KMAs acting as dynamical symmetries to be distinguished from $SU(2)$ KMA symmetry and from fundamental KMA symmetries assignable to the isometries and holonomies of $M^4 \times CP_2$.

One can of course ask whether also the fundamental symmetries could have a representation in terms of Gal or its semi-direct product $G \triangleleft Gal$ with a finite sub-group automorphism group $SO(3)$ of quaternions lifting to finite subgroup $G \subset SU(2)$ acting on spinors. This is not necessary since Gal can form semidirect products with the algebraic subgroups of Lie groups of fundamental symmetries (Langlands program relies on this). In the generic case the algebraic subgroups spanned by given extension of rationals are infinite. When the finite subgroup $G \subset SU(2)$ is closed under Gal automorphism, the situation changes, and these extensions are expected to be in a special role physically.

The number theoretic generalization of the idea that affine ADE group acts as symmetries would be roughly like following. The nodes of the McKay graph of $G \triangleleft Gal$ label its irreps, which should be in 1-1 correspondence with the Cartan algebra of the KMA. The KMA counterparts of the local bilinear Gal invariants associated with Gal irreps would give currents of dynamical KMA having unit conformal weight. The convolution of primary fields with respect to conformal weight would be completely analogous to that occurring in the expression of energy momentum tensor as local bilinears of KMA currents.

If the free field construction using the local invariants as Cartan algebra defined by the irreps of $G \triangleleft Gal$ works, it gives rise to charged primary fields for the dynamical KMA labelled by roots of the corresponding Lie algebra. For trivial Gal one would have ADE group acting as dynamical symmetries of minimal model associated with $G \subset SU(2)$.

5. Number theoretic Langlands conjecture [L2] [K22] generalizes this to the semidirect product $G_0 \triangleleft Gal$ algebraic subgroup G_0 of the original KMA Lie group (p-adicization allows also powers of roots of e in extension). One can imagine a hierarchy of KMA type algebras KMA_n obtained by repeating the procedure for the $G_1 \triangleleft Gal$, where G_1 is discrete subgroup of the new KMA Lie group.

6. In CFTs are also other manners to extend VA or SVA (Super-Virasoro algebra) to a larger algebra by discovering new dynamical symmetries. The hope is that symmetries would allow to solve the CFT in question. The general constraint is that the conformal weights of symmetry generators are integer or half-integer valued. For the energy momentum tensor defining VA the conformal weight is $h = 2$ whereas the conformal weights of primary fields for minimal models are rational numbers.

   The simplest extension is SVA involving super generators with $h = 3/2$. Extension of (S)VA by (S)KMA so that (S)VA acts by semidirect product on (S)KMA means adding (S)KMA generators with with $h = 1$ (and 1/2). The generators of W_n-algebras (see `http://tinyurl.com/y93f6eoo`) have either integer or half integer conformal weights and the algebraic operations are defined as ordered products (an associative operation). These extensions are different from the proposed number theoretic extension for which the restriction to a discrete subgroup of KMA Lie group is essential.

5 Appendix

I have left the TGD counterpart of *fake flatness condition* in Appendix. Also a little TGD glossary is included.

5.1 What could be the counterpart of the fake flatness in TGD framework?

Schreiber considers the n-variant of gauge field concept with gauge potential A and gauge field $F = DA$ replaced with a hierarchy of gauge potential like entities $A^{k)}$, $k = 1, .., n$ with $DA^{n)} = 0$ and ends up in $n = 2$ case to what he calls *fake flatness condition* $DA^{1)} = A^{2)}$. This raises a chain of questions.

Could higher gauge fields of Schreiber and Baez [B5, B4] provide a proper description of the situation in finite measurement resolution? Could induction procedure make sense for them? Should one define the projections of the classical fields by replacing ordinary H-coordinates with their quantum counterparts? Could these reduce to c-numbers for number-theoretically commutative

2-surfaces with commutative tangent space? What about second fundamental form orthogonal to the string world sheet? Must its trace vanish so that one would have minimal 2-surface?

The proposal of Schreiber is a generalization of a massless gauge theory. My gut feeling is that the non-commutative counterpart of space-time surface is not promising in TGD framework. My feelings are however mixed.

1. The effective reduction of SSA and PSCA to quantal variants of Kac-Moody algebras gives rise to a theory much more complex than gauge theory. On the other hand, the reduction to Galois groups by finiteness of measurement resolution would paradoxically reduce TGD to extremely simple theory.

2. Analog of Yang Mills theory is not enough since it describes massless particles. Massless states in 4-D sense are only a very small portion of the spectrum of states in TGD. Stringy mass squared spectrum characterizes these theories rather than massless spectrum. On the other hand, in TGD particles are massless in 8-D sense and this is crucial for the success of generalized twistor approach.

3. To define a generalization of gauge theory in WCW one needs homology and cohomology for differential forms and their duals. For infinite-dimensional WCW the notion of dual is difficult to define. The effective reduction of SSA and PSCA to SKMAs could however effectively replace WCW with a coset space of the Lie-group associated with SKMA and finite dimension would allow tod define dual.

4. The idea about non-Abelian counterparts of Kähler gauge potential A and J in WCW does not look promising and the TGD counterpart of the *fake flatness condition* does not however encourage this.

Just for curiosity one could however ask whether one could generalize the Kähler structure of WCW to n-Kähler structure to describe finite measurement resolution at the level of WCW and whether also now something analogous to the *fake flatness condition* emerges. The "fake flatness" condition has interesting analogy in TGD framework starting from totally different geometric vision.

1. SSA acts as dynamical symmetries on fermions at string world sheets. Gauge condition would make the situation effectively finite-dimensional and allow to define if the effectively finite-D variant of WCW n-structures using ordinary homotopies and homology and cohomology. Also n-gauge fields could be defined in this effectively finite-D WCW and they would allow a description in terms of string world sheets. The interpretation could be in terms of generalization of Bohm-Aharonov phase from space-time level to Berry phase in abstract configuration space defined now in reduced WCW.

2. The Kähler form of $H = M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) can be induced to space-time level. When limited to the string world sheet is both the curvature form of Kähler potential and the analog of flat 2-connection defining the 1-connection in the approaches of Schreiber's and Baez so that one would have $B = J$ and $dB = 0$.

3. 2-form J as it is interpreted in Screiber's approach is hwoever not enough to construct WCW geometry. The generalization of the geometry of $M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) to involve higher forms and its induction to the space-time level and level of WCW looks rather awkward idea and does not bring in anything new.

5.2 A little glossary

Topological Geometrodynamics (TGD): TGD can be regarded as a unified theory of fundamental interactions. In General Relativity space-time time is abstract pseudo-Riemannian manifold and the dynamical metric of the space-time describes gravitational interactions. In TGD space-time is a 4-dimensional surface of certain 8-dimensional space, which is non-dynamical and fixed by either physical or purely mathematical requirements. Hence space-time has shape besides metric properties. This identification solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity. Even more, sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry behind General Relativity, leads to a geometrization of known fundamental interactions and elementary particle quantum numbers.

Many-sheeted space-time, topological quantization, quantum classical correspondence (QCC): TGD forces the notion of many-sheeted space-time (see `http://tinyurl.com/mf99gpv`) with space-time sheets identified as geometric correlates of various physical objects (elementary particles, atoms, molecules, cells, ..., galaxies, ...). Quantum-classical correspondence (QCC) states that all quantum notions have topological correlates at the level of many-sheeted space-time.

Topological quantization: Topological field quantization is one of the basic distinctions between TGD and Maxwell's electrodynamics and GRT and means that various fields decompose to topological field quanta: radiation fields to "topological light rays"; magnetic fields to flux tube structures; and electric fields to electric flux quanta (electrets). Topological field quantization means that one can assign to every material system a field (magnetic) body, usually much larger than the material system itself, and providing a representation for various quantum aspects of the system.

Strong form of holography (SH): SH states that space-time surfaces as preferred extremals can be constructed from the data given at 2-D string world sheets and by a discrete set of points defining the cognitive representation of the space-time surface as points common to real and various p-adic variants of the space-time surface (intersection of realities and various p-adicities). Points of the cognitive representation have imbedding space coordinates in the extension of rationals defining the adele in question. Effective 2-dimensionality is a direct analogy for the continuation of 2-D data to analytic function of two complex variables.

Zero energy ontology (ZEO): In ZEO quantum states are replaced by pairs of positive and negative energy states having opposite total quantum numbers. The pair corresponds to the pair of initial and final state for a physical event in classical sense. The members of the pair are at opposite boundaries of causal diamond (CD) (see `http://tinyurl.com/mh9pbay`), which is intersection of future and past directed light-cones with each point replaced with CP_2. Given CD can be regarded as a correlate for the perceptive field of conscious entity.

p-Adic physics, adelic physics, hierarchy of Planck constants, p-adic length scale hypothesis: p-Adic physics is a generalization of real number based physics to p-adic number fields and interpreted as a correlate for cognitive representations and imagination. Adelic physics fuses real physics with various p-adic physics ($p = 2, 3, 5, ...$) to adelic physics. Adele is structure formed by reals and extensions of various p-adic number fields induced by extensions of rationals forming an evolutionary hierarchy. Hierarchy of Planck constants corresponds to the hierarchy of orders of Galois groups for these extensions. Preferred p-adic primes satisfying p-adic length scale hypothesis $p \simeq 2^k$, are so called ramified primes for certain extension of rationals appearing as winners in algebraic evolution.

Cognitive representation: Cognitive representation corresponds to the intersection of the sensory and cognitive worlds - realities and p-adicities - defined by real and p-adic space-time surfaces. The points of the cognitive representation have H-coordinates which belong to an extension of rationals defining the adele. The choice of H-coordinates is in principle free but symmetries of H define preferred coordinates especially suitable for cognitive representations. The Galois group of the extension of rationals has natural action in the cognitive representation, and one can decompose it into orbits, whose points correspond the sheets of space-time surface as Galois covering. The number n of sheets equals to the dimension of the Galois group in the general case and is identified as the value $h_{eff}/h = n$ of effective Planck constant characterizing levels in the dark matter hierarchy. One can also consider replacing space-time surfaces as points of WCW with their cognitive representations defined by the cognitive representation of the space-time surface and defining the natural coordinates of WCW point.

Quantum entanglement, negentropic entanglement (NE), Negentropy maximization principle (NMP): Quantum entanglement does not allow any concretization in terms of everyday concepts. Schrödinger cat is the classical popularization of the notion (see `http://tinyurl.com/lpjcjm9`): the quantum state, which is a superposition of the living cat + the open bottle of poison and the dead cat + the closed bottle of poison represents quantum entangled state. Schrödinger cat has clearly no self identity in this state.

In adelic physics one can assign to the same entanglement both real entropy and various p-adic negentropies identified as measures of conscious information. p-Adic negentropy - unlike real - can be positive, and one can speak of negentropic entanglement (NE). Negentropy Maximization Principle (NMP) states that it tends to increase. In the adelic formulation NMP holding true only in statistical sense is a consequence rather than separate postulate.

Self, subself, self hierarchy: In ZEO self is generalized Zeno effect. At the passive boundary nothing happens to the members of state pairs and the boundary remains unaffected. At active boundary members of state pairs change and boundary itself moves farther away from the passive boundary reduction by reduction inducing localization of the active boundary in the moduli space of CDs after unitary evolution inducing delocalization in it. Self dies as the first reduction takes place at opposite boundary. A self hierarchy extending from elementary particle level to the level of the entire Universe is predicted. Selves can have sub-selves which they experience as mental images. Sub-selves of two separate selves can quantum entangle and this gives rise to fusion of the mental images and the fused mental image is shared by both selves.

Sensory representations: The separation of data processing and its representation is highly desirable. In computers processing of the data is performed inside CPU and representation is realized outside it (monitor screen, printer,...). In standard neuroscience it is however believed that both data processing and representations are realized inside brain. TGD leads the separation of data processing and representations: the "manual" of the material body provided by field (or magnetic) body serves as the counterpart of the computer screen at which the sensory and other representations of the data processed in brain are realized. Various attributes of the objects of the perceptive field processed by brain are quantum entangled with simple "something exists" mental images at the MB. The topological rays of EEG serve are the electromagnetic bridges serving as the topological correlates for this entanglement.

REFERENCES

Mathematics

[A1] Connes A. *Non-commutative Geometry.* Academic Press, San Diego, 1994.

[A2] Whitten E. Resolution of singularities in algebraic varieties. Available at:`http://tinyurl.com/yb7cuwkf`, 2008.

[A3] McKay J. Cartan matrices, finite groups of quaternions, and kleinian singularities. *Proc AMS.* *`http://tinyurl.com/ydygjgge`*, 1981.

[A4] Reid M. The du val singularities an, dn, e6, e7, e8. Available at:`http://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf`.

[A5] Reid M. Chapters on algebraic surfaces. Available at:`https://arxiv.org/abs/alg-geom/9602006`, 1996.

[A6] Jones V. In and around the origin of quantum groups. Available at: `http://arxiv.org/abs/math/0309199`, 2003.

Theoretical Physics

[B1] The quantum super yangian and casimir operators of $u_q(gl(m|n))$. arXiv:hep-th/9407135. Available at:`http://tinyurl.com/l8asm33`, 1994.

[B2] Zuber J-B Cappelli A. A-D-E Classification of Conformal Field Theories. Available at:`http://tinyurl.com/h9val5g`, 2010.

[B3] Yang-Hui He. On algebraic singularities, finite graphs and d-brane gauge theories: A string theoretic perspective. Available at arXiv:hep-th/0209230. `https://arxiv.org/pdf/hep-th/0209230.pdf`, 2002.

[B4] Baez JC. Higher yang-mills theory. arXiv:hep-th/0206130. Available at:http://xxx.lanl.gov/abs/hep-th/0206130, 2002.

[B5] Schreiber U. Nonabelian 2-forms and loop space connections from scft deformations. arXiv:hep-th/0407122. Available at:https://arxiv.org/abs/hep-th/0407122, 2004.

Books related to TGD

[K1] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html#elvafu, 2006.

[K2] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdgeom.html#compl1, 2006.

[K3] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Hyper-finite Factors and Dark Matter Hierarchy*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/neuplanck.html#vNeumannnew, 2006.

[K4] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdgeom.html#kahler, 2006.

[K5] Pitkänen M. Langlands Program and TGD. In *TGD as a Generalized Number Theory*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#langlands, 2006.

[K6] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html#mless, 2006.

[K7] Pitkänen M. New Particle Physics Predicted by TGD: Part I. In *p-Adic Physics*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html#mass4, 2006.

[K8] Pitkänen M. New Particle Physics Predicted by TGD: Part II. In *p-Adic Physics*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html#mass5, 2006.

[K9] Pitkänen M. Quantum Hall effect and Hierarchy of Planck Constants. In *Hyper-finite Factors and Dark Matter Hierarchy*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/neuplanck.html#anyontgd, 2006.

[K10] Pitkänen M. *Quantum Physics as Infinite-Dimensional Geometry*. Online book.Available at: http://www.tgdtheory.fi/tgdhtml/tgdgeom.html, 2006.

[K11] Pitkänen M. *TGD as a Generalized Number Theory*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html, 2006.

[K12] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#visionb, 2006.

[K13] Pitkänen M. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdclass.html#tgdgrt, 2006.

[K14] Pitkänen M. Was von Neumann Right After All? In *Hyper-finite Factors and Dark Matter Hierarchy*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html#vNeumann, 2006.

[K15] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdgeom.html#cspin, 2006.

[K16] Pitkänen M. Construction of Quantum Theory: More about Matrices. In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#UandM, 2012.

[K17] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. In *Hyper-finite Factors and Dark Matter Hierarchy.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/neuplanck.html#tensornet, 2014.

[K18] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdgeom.html#wcwnew, 2014.

[K19] Pitkänen M. Unified Number Theoretical Vision. In *TGD as a Generalized Number Theory.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#numbervision, 2014.

[K20] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#fermizeta, 2015.

[K21] Pitkänen M. About twistor lift of TGD? In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#hgrtwistor, 2016.

[K22] Pitkänen M. Langlands Program and TGD: Years Later. In *TGD as a Generalized Number Theory.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#langlandsnew, 2016.

[K23] Pitkänen M. The classical part of the twistor story. In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#twistorstory, 2016.

[K24] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#Yangianagain, 2017.

[K25] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdnumber.html#adelephysics, 2017.

[K26] Pitkänen M. Some questions related to the twistor lift of TGD. In *Towards M-Matrix.* Online book. Available at: http://www.tgdtheory.fi/tgdhtml/tgdquantum.html#twistquestions, 2017.

Articles about TGD

[L1] Pitkänen M. About some unclear issues of TGD. Available at: http://tgdtheory.fi/public_html/articles/fermionsector.pdf, 2017.

[L2] Pitkänen M. Are higher structures needed in the categorification of TGD? Available at: http://tgdtheory.fi/public_html/articles/nstructures.pdf, 2017.

[L3] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? Available at: http://tgdtheory.fi/public_html/articles/Yangianagain.pdf, 2017.

[L4] Pitkänen M. p-Adicization and adelic physics. Available at: http://tgdtheory.fi/public_html/articles/adelicphysics.pdf, 2017.

[L5] Pitkänen M. Philosophy of Adelic Physics. Available at: http://tgdtheory.fi/public_html/articles/adelephysics.pdf, 2017.

[L6] Pitkänen M. Re-examination of the basic notions of TGD inspired theory of consciousness. Available at: http://tgdtheory.fi/public_html/articles/conscrit.pdf, 2017.

[L7] Pitkänen M. What could be the role of complexity theory in TGD? Available at: http://tgdtheory.fi/public_html/articles/numberchaos.pdf, 2017.