

# Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part I

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December 30, 2019

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**Abstract**

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called  $M^8 - H$  duality is one of these approaches. The beauty of  $M^8 - H$  duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.
2. It will be shown how  $M^8 - H$  duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in  $M^8$  would be algebraic surfaces identified as zero loci for imaginary part  $IM(P)$  or real part  $RE(P)$  of octonionic polynomial of complexified octonionic variable  $o_c$  decomposing as  $o_c = q_c^1 + q_c^2 I^4$  and projected to a Minkowskian sub-space  $M^8$  of complexified  $O$ . Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm  $q_c \bar{q}_c$  appearing in  $RE(P)$  or  $IM(P)$  caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for quaternions at light-cone boundaries.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part  $RE(P)$  (imaginary parts  $IM(P)$ ).  $RE(P)$  and  $IM(P)$  are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification  $M^4 \subset O$  as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and  $M^8 - H$  correspondence could generalize.

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates  $RE(Y)^i$  or  $IM(Y)^i$  in the decomposition  $Y^i = RE(Y)^i + IM(Y)^i I_4$  of the gradient of  $RE(P) = Y = 0$  with respect to the complex coordinates  $z_i^k$ ,  $k = 1, 2$ , of  $O$  vanishes that is critical as function of quaternionic components  $z_1^k$  or  $z_2^k$  associated with  $q_1$  and  $q_2$  in the decomposition  $o = q_1 + q_2 I_4$ , call this component  $X_i$ . In the generic case this gives 3-D surface.

In this generic case  $M^8 - H$  duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to  $H$ , and only determines the boundary conditions of the

dynamics in  $H$  determined by the twistor lift of Kähler action.  $M^8 - H$  duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial  $P$  so that the criticality conditions do not reduce the dimension:  $X_i$  would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components  $X_i$ . Space-time surface would be analogous to a polynomial with a multiple root. The criticality of  $X_i$  conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in  $H$  in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by  $M^8 - H$  duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles.  $M^8 - H$  duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics.  $M^8 - H$  duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

## 1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

### 1.1 Various approaches to classical TGD

#### 1.1.1 World of classical worlds

The first approach is based on the geometry of the "world of classical worlds" (WCW) [K3, K2, K10].

1. The study of classical field equations led rather early to the realization that preferred extremals for the twistor lift of Kähler action with Minkowskian signature of induced metric define a slicing of space-time surfaces defined by 2-D string world sheets and partonic two-surfaces locally orthogonal to them. The interpretation is in terms of position dependent light-like momentum vector and polarization vector defining the local decompositions  $M^2(x) \times E^2(x)$  of tangent space integrating to a foliation by partonic 2-surfaces and string world sheets. I christened this structure Hamilton-Jacobi structure. Its Euclidian counterpart is complex structure in Euclidian regions of space-time surface.
2. The formulation of quantum TGD in terms of spinor fields in WCW [K9] leads to the conclusion that WCW must have Kähler geometry [K3, K2] and has it only if it has maximal

group of isometries identified as symplectic transformations of  $\delta M_{\pm}^4 \times CP_2$ , where  $\delta M_{\pm}^4$  denotes light cone boundary two which upper/lower boundary of causal diamond (CD) belongs. Symplectic Lie algebra extends naturally to supersymplectic algebra (SSA).

3. Space-time surfaces would be preferred extremals of twistor lift of Kähler action [K19] and the conditions realizing strong form of holography (SH) would state that sub-algebra of SSA isomorphic with it and its commutator with SSA give rise to vanishing Noether charges and these charges annihilate physical states or create zero norm states from them. One should solve these conditions.
4. The dynamics involves also fermions. Induced spinor fields are located inside space-time surface but for some yet not completely understood reason only the information about spinor at 2-D string world sheets is needed in the construction of scattering amplitudes. This dynamics would be 2-dimensional. The construction of twistor amplitudes even suggests that it is 1-dimensional in the sense that 1-D light-like curves at light-like partonic orbits defining boundaries of Minkowskian and Euclidian regions determines the scattering amplitudes. String world sheets are however needed only as correlates for entanglement between fermions at different partonic orbits.

The 2-D character of fermionic dynamics conforms with the strong form of holography (SH) but how the string world sheets and partonic 2-surfaces are selected from Hamilton-Jacobi slicing? Electromagnetic neutrality could select string worlds sheets but one can actually always find a gauge in which the induced classical electroweak field at these surfaces is purely electromagnetic.

### 1.1.2 Twistor lift of TGD

Second approach to preferred extremals is based on TGD version [K16, K15, K14, K19] of twistor Grassmann approach [B1, B4, B3].

1. The twistor lift of TGD leads to a proposal that space-time surfaces can be represented as sections in their 6-D twistor spaces identified as twistor bundles in the product  $T(H) = T(M^4) \times T(CP_2)$  of 6-D twistor spaces of  $M^4$  and  $CP_2$ . Twistor structure would be induced from  $T(H)$ . Kähler action can be lifted to the level of twistor spaces only for  $M^4 \times CP_2$  since only for these spaces twistor space allows Kähler structure [A2]. Twistors were originally introduced by Penrose with the motivation that one could apply algebraic geometry in Minkowskian signature. The bundle property is extremely powerful and should be consistent with the algebraic geometrization at the level of  $M_c^8$ . The challenge is to formulate the twistor lift at the level of  $M^8$ .
2. The twistor lift of Kähler action contains also volume term. Field equations have two kinds of solutions. For the solutions of first kind the dynamics of volume term and Kähler action are coupled and the interpretation is in terms of interaction regions. Solutions of second kind are minimal surfaces and extremals of both Kähler action and volume term, whose dynamics decouple completely and all coupling constants disappear from the dynamics. These extremals are natural candidates for external particles. For these solutions at least the field equations reduce to the existence of Hamilton-Jacobi structure. The completely universal dynamics of these regions suggests interpretation in terms of maximal quantum criticality characterized by the extension of the usual conformal invariance to its quaternionic analog.
3. A connection with zero energy ontology (ZEO) emerges. Causal diamond (CD, intersection of future and past directed light-cones of  $M^4$  with points replaced by  $CP_2$ ) would naturally determine the interaction region to which external particles enter through its 2 future and past boundaries. But where does ZEO emerge?

### 1.1.3 $M^8 - Hduality$

The third approach is based on number theoretic vision [K7, K8, K6, K11].

1.  $M^8 - H$  duality [K8, K11, K12] means that one can see space-times as 4-surfaces in either  $M^8$  or  $H = M^4 \times CP_2$ . One could speak “number theoretical compactification” having however nothing to do with stringy version of compactification, which is dynamical.  $M^8 - H$  duality suggests that space-time surfaces in  $H = M^4 \times CP_2$  are images of space-time surfaces in  $M^8$  or actually of  $M^8$  projections of complexified space-time surfaces in  $M_c^8$  identified as space of complexified octonions. These space-time surfaces could contain the integrated distributions of string world sheets and partonic 2-surfaces mentioned in the previous item. Space-time surfaces must have associative tangent or normal space for  $M^8 - H$  correspondence to exist.
2. The fascinating possibility mentioned already earlier is that in  $M^8$  these surfaces could correspond to zero loci for real or imaginary parts of real analytic octonionic polynomials  $P(o) = RE(P) + IM(P)I_4$ ,  $I_4$  an octonionic imaginary unit orthogonal to quaternionic ones. The condition  $IM(P) = 0$  ( $RE(P) = 0$ ) would give associative (co-associative) space-time surface. In the simplest case these functions would be polynomials so that one would have algebraic geometry for algebraically 4-D complex surfaces in 8-D complex space.

**Remark:** The naive guess that space-time surfaces reduce to quaternionic curves in quaternionic plane fails due to the non-commutativity of quaternions meaning that one has  $P(o) = P(q_1, q_2, \bar{q}_1, \bar{q}_2)$  rather than  $P(o) = P(q_1, q_2)$ .

**Remark:** Why not rational functions expressible as ratios  $R = P_1/P_2$  of octonionic polynomials? It has become clear that one can develop physical arguments in favor of this option. The zero loci for  $IM(P_i)$  would represent space-time varieties. Zero loci for  $RE(P_1/P_2) = 0$  and  $RE(P_1/P_2) = \infty$  would represent their interaction presumably realized as wormhole contacts connecting these varieties. In the sequel most considerations are for polynomials: the replacement of polynomials with rational functions does not introduce big differences and its discussed in the section “Gromov-Witten invariants, Riemann-Roch theorem, and Atiyah-Singer index theorem from TGD point of view” of [L5].

3. The objection against this proposal is obvious.  $M^8 - H$  correspondence cannot hold true since the dynamics defined by octonionic polynomials in  $M^8$  contains no coupling constants whereas the dynamics of twistor lift of Kähler action depends on coupling constants in the generic space-time region. However, for space-time surfaces representing external particles entering inside CD at its boundaries this is however not the case! They could satisfy  $M^8 - H$  correspondence!

This suggests that inside CDs the space-time surfaces are not associative/co-associative in  $M^8$  so that  $M^8 - H$  correspondence cannot map them to  $H$  and the twistor lifted Kähler action and SH take care of the dynamics. External particles are associative and quantum critical and  $M^8 - H$  correspondence makes sense. The quantum criticality and associativity at the boundaries of CD poses extremely powerful conditions and allows to satisfy infinite number of vanishing conditions for SSA charges.

It has later turned out [L9] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

4. This picture is consistent with the the explicit formulation of the associativity conditions  $Re(P) = 0$  and  $IM(P) = 0$  for varieties. The calculations shows that associativity can be realized either by posing a condition making them 3-dimensional except, when the situation is critical in the sense that the 4-D variety is analogous to a double root of polynomial: now however the polynomial corresponds to prime polynomial decomposing to product of polynomials in the extension of rationals such that the product contains higher powers of the factors. One has ramification at the level of polynomial primes so that the criticality condition does not bring anything new but need not make the situation associative. At most 3 conditions need to be applied to guarantee associativity and they might leave the space-time surface 4-D.

5. The coordinates of  $M^4$  as octonionic roots  $x + iy$  of the 4 real polynomials need not be real. Space-time surface must have  $M_c^4$  projection, which reduces to  $M^4$ . There are two options.

(a) The original proposal was that the *projection* from  $M_c^8$  to real  $M^4$  (for which  $M^1$  coordinate is real and  $E^3$  coordinates are imaginary with respect to  $i$ !) defines the real space-time surface mappable by  $M^8 - H$  duality to  $CP_2$ . One can however criticize the allowance of a nonvanishing imaginary part of space-time surface in  $M_c^4$ .

(b) A more stringent condition is that the roots of the 4 vanishing polynomials as coordinates of  $M_c^4$  belong automatically to  $M^4$  so that  $m^0$  would be real root and  $m^k$ ,  $k = 1, \dots, 3$  imaginary with respect to  $i \rightarrow -i$ .  $M_c^8$  coordinates would be invariant ("real") under combined conjugation  $i \rightarrow -i, I_k \rightarrow -I_k$ . In the following I will speak about this property as *Minkowskian reality*.

This could allow to identify CDs in very elegant manner: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

6. This octonionic view as also lower-dimensional quaternionic counterpart. In this case one considers 2-D commutative/co-commutative surfaces tentatively identifiable as string world sheets and partonic 2-surfaces. Why not all 2-surfaces appearing in the Hamilton-Jacobi slicing are not selected? The above mechanism would work also now. The commutativity conditions reduce in the generic case give 1-D curve as a solution. The interpretation would be as orbit of point like particle at 3-D partonic orbit appearing in the construction of twistorial amplitudes. In critical situation one would obtain string world sheet serving as a correlate for entanglement between point like particles at its ends: one would have quantum critical bound state.

I have considered also other attempts to define what quaternion structure could mean.

1. One could also consider the possibility that the tangent spaces of space-time surfaces in  $H$  are associative or co-associative [K11]. This is not necessary although it seems that this might be the case for the known extremals. If this holds true, one can construct further preferred extremals by functional composition by generalization of  $M^8 - H$  correspondence to  $H - H$  correspondence.

2. I have considered also the possibility of quaternion analyticity in the sense of generalization of Cauchy-Riemann equations, which tell that left- or right quaternionic differentiation makes sense [L3]. It however seems that this approach is not promising. The conditions are quite too restrictive and bring nothing essentially new. Octonion/quaternion analyticity in the above mentioned sense does not require the analogs of Cauchy-Riemann conditions.

## 1.2 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.

2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by  $CP_2$  points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces

an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.

3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L1]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials  $P(o)$  containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with  $RE(P) = 0$  can transform to  $IM(P) = 0$  region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.
4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adèle for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.
5. Also a connection with infinite primes is suggestive [K8]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to  $M^8 - H$  duality. The strategy is simple: try to remember all previous objections against  $M^8 - H$  duality and invent new ones since this is the best manner to make real progress.

## 1.3 Topics to be discussed

### 1.3.1 Key notions and ideas of algebraic geometry

Before going of octonionic algebraic geometry, I will discuss basic notions of algebraic geometry such as algebraic variety (see <http://tinyurl.com/hl6sjmz>), - surface (see <http://tinyurl.com/y8d5wsmj>), and - curve (see <http://tinyurl.com/nt6tkey>), rational point of variety central for TGD view about cognitive representation, elliptic curves (see <http://tinyurl.com/lovksny>) and - surfaces (see <http://tinyurl.com/yc33a6dg>), and rational points (see <http://tinyurl.com/ybbnysu>) and potentially rational varieties (see <http://tinyurl.com/yabl4xt>). Also the notion of Zariski topology (see <http://tinyurl.com/h5pv4vk>) and Kodaira dimension (see <http://tinyurl.com/yadoj2ut>) are discussed briefly. I am not a mathematician. What hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

Much of algebraic geometry is counting numbers of say rational points or of varieties satisfying some conditions. One can also count dimensions of moduli spaces. Hence the basic notions and methods of enumerative geometry are discussed. There is also a discussion of Gromow-Witten invariants and Riemann-Roch theorem having Atiyah-Singer index theorem as a generalization. These notions will be applied in the second part of the article [L5].

### 1.3.2 $M^8 - H$ duality

$M^8 - H$  duality [K12, K8, K11] would reduce classical TGD to the algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. Space-time surfaces in  $M^8$  would be algebraic varieties identified as zero loci



for imaginary part  $IM(P)$  or real part  $RE(P)$  of octonionic polynomial of complexified octonionic variable  $o$  decomposing as  $o = q_c^1 + q_c^2 I_4$  and projected to a Minkowskian sub-space  $M^8$  of  $o$ . Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces in  $M^8$  would form commutative and associative algebra with addition, product and functional composition.

As already noticed, the associativity conditions do not allow 4-D solutions except for criticality so that  $M^8 - H$  correspondence can hold true only in these space-time regions and one has these nice features at the level of  $M^8$ . In critical regions  $M^8 - H$  correspondence is true and these features have  $H$  counterparts

The basic problem is to understand the map mediating  $M^8 - H$  duality mapping the point  $(m, e)$  of  $M^8 = M_0^4 \times E^4$  to a point  $(m, s)$  of  $M_0^4 \times CP_2$ , where  $M_0^4$  point is obtained as a projection to a suitably chosen  $M_0^4 \subset M^8$  and  $CP_2$  point parameterizes the tangent space as quaternionic sub-space containing preferred  $M_0^2(x) \subset M^4(x)$ . This map involves slightly non-local information and could allow to understand why the preferred extremals at the level of  $H$  are determined by partial differential equations rather than algebraic equations. Also the generalization to the level of twistor lift is briefly touched.

### 1.3.3 Challenges of the octonionic algebraic geometry

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part  $RE(P)$  (imaginary parts  $IM(P)$ ).  $RE(P)$  and  $IM(P)$  are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification  $M^4 \subset O$  as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction (see <http://tinyurl.com/ybuy1a2k>) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a  $M^8 - H$  correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates  $RE(Y)^i$  or  $IM(Y)^i$  in the decomposition  $Y^i = RE(Y)^i + IM(Y)^i I_4$  of the gradient of  $RE(P) = Y = 0$  with respect to the complex coordinates  $z_i^k$ ,  $k = 1, 2$ , of  $O$  vanishes that is critical as function of quaternionic components  $z_1^k$  or  $z_2^k$  associated with  $q_1$  and  $q_2$  in the decomposition  $o = q_1 + q_2 I_4$ , call this component  $X_i$ . In the generic case this gives 3-D surface.

In this generic case  $M^8 - H$  duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to  $H$ , and only determines the boundary conditions of the dynamics in  $H$  determined by the twistor lift of Kähler action.  $M^8 - H$  duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial  $P$  so that the criticality conditions do not reduce the dimension:  $X_i$  would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components  $X_i$ . Space-time surface would be analogous to a polynomial with a multiple root. The criticality of  $X_i$  conforms with

the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A1] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in  $H$  in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by  $M^8 - H$  duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles.  $M^8 - H$  duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics.  $M^8 - H$  duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

The easiest manner to kill  $M^8 - H$  duality in the form it is represented here is to prove that 4-D zero loci for imaginary/real parts of octonionic polynomials with real coefficients can never be associative/co-associative being always 3-D. I hope that some professional mathematician would bother to check this.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

## 2 Some basic notions, ideas, results, and conjectures of algebraic geometry

In this section I will summarize very briefly the basic notions of algebraic geometry needed in the sequel.

### 2.1 Algebraic varieties, curves and surfaces

The basic notion of algebraic geometry is algebraic variety.

1. One considers affine space  $A^n$  with  $n$  coordinates  $x^1, \dots, x^n$  having values in a number field  $K$  usually assumed to be algebraically closed (note that affine space has no preferred origin like linear space). Algebraic variety is defined as a solution of one or more algebraic equations stating the vanishing of polynomials of  $n$  variables:  $P^i(x^1, \dots, x^n) = 0$ ,  $i = 1, \dots, r \leq n$ . One can restrict the coefficients of polynomials to p-adic number field or or its extension to an extension of rationals. One talks about polynomials on  $k \subset K$ .
2. The basic condition is that the variety is not a union of disjoint varieties. This for instance happens, when the polynomial  $P(x^1, \dots, x^n)$  defining co-dimension 1 manifold is product of polynomials  $P = \prod_r P_r$ . Algebraic variety need not be a manifold meaning that it can have

singular points. For instance, for co-dimension 1 variety the Jacobian matrix  $\partial P/\partial x^i$  of the polynomial can vanish at singularity.

3. One can define projective varieties (see <http://tinyurl.com/ybsqvy3r>) in projective space  $P^n$  having coordinatization in terms of  $n+1$  homogenous coordinates  $(x^1, \dots, x^{n+1})$  in  $K$  with points differing by an overall scaling identified. Projective variety is defined as zero locus of homogenous polynomials of  $n+1$  coordinates so that solutions remain solutions under the overall scaling of all coordinates. By identifying the points related by scaling one obtains a surface in  $P^n$ . Grassmannian of linear space  $V^n$  (not affine space!) is a projective spaces defined as space of  $k$ -planes of  $V^n$ . These spaces are encountered in twistor Grassmannian approach to scattering amplitudes.

For polynomials of single variable one obtains just the roots of  $P_n(x) = 0$  in an algebraic extension assignable to the polynomial. For several variables one can in principle proceed step by step by solving variable  $x^1$  as algebraic function of others from  $P_1(x^1, \dots, x^n) = 0$ , proceed to solve  $x^2$  from  $P_2(x^1(x^2, \dots), x^2, \dots) = 0$  as an algebraic function of the remaining variables, and so on. The algebraic functions involved get increasingly complex but in some exceptional situations the solution has parametric representation in terms of *rational* rather than algebraic functions of parameters  $t^k$ . For co-dimension  $d_c > 1$  case the intersection of surfaces  $P^i = 0$  need not be complete and the tangent spaces of the hyper-surfaces  $P^i = 0$  need not intersect transversally in the generic case. Therefore  $d_c > 1$  case is not gained so much attention as  $d_c = 1$  case.

A more advanced treatment relies on ring theory by assigning to polynomials a ring as the ring of polynomials in the space involved divided by the ring of polynomials vanishing at zero loci of polynomials  $P^i$ .

1. The notion of ideal is central and determined as a subring invariant under the multiplication by elements of ring. Prime ideal generalizes the notion of prime and one can say that the notion of integer generalizes to that of ideal. One can also define the notion of fractional ideal.
2. Zariski topology (see <http://tinyurl.com/h5pv4vk>) replacing the topology based on real norm is second highly advanced notion. The closed sets in this topology are algebraic varieties of various dimensions. Since the complement of any algebraic variety is open set this topology and open also in the ordinary real topology, this topology is considerable rougher than the ordinary topology.

Some remarks from the point of view of TGD are in order.

1. In the scenario inspired by  $M^8 - H$  duality one has co-dimension 4 surfaces in 8-D complex space. Octonionicity of polynomials however implies huge symmetries since the polynomial is determined by single real polynomial of real variable, whose values at finite number of points determined the polynomial.
2. In TGD the extension of rationals can be assumed to contain also powers for some root of  $e$  since in p-adic context this gives rise to a finite-dimensional extensions due to the fact that  $e^p$  is ordinary p-adic number. Also a restriction to a finite field are possible and restriction of rational coefficients to their modulo  $p$  counterparts reduces the polynomial to polynomial in finite field. This reduction is used as a technical tool. In the case of Diophantine equations (see <http://tinyurl.com/nt6tkey> and <http://tinyurl.com/y8hm4zce>) the coefficients are restricted to be integers.
3. In adelic TGD [L7] [L6] the number fields involved are reals and extensions of p-adic numbers. The coefficient field for the coefficients of polynomials would be naturally extension of rationals or extension of p-adics induced by it. The coefficients of polynomials serve as coordinates of adelic WCW. p-Adic numbers are not algebraically closed and one must assume an extension of p-adic numbers from that for the coefficients one to allow maximal number of roots.

This suggests an evolutionary process [L8] extending the number field for the coefficients of polynomials. Arbitrary root of polynomial for given extension can be realized only if the

original extension is extended further. But this allows polynomial coefficients in this new extension: WCW is now larger. Now one has however roots in even larger extension so that the unavoidable outcome is number theoretic evolution as increase of complexity.

4. What is so remarkable is that octonionic polynomials with rational coefficients could be determined by their values at finite set of points for a polynomial of real argument once the order of polynomial is fixed. Real coordinate corresponds to preferred time axis naturally. A cognitive representation consisting of finite number of rational points could fix the entire space-time surface! This would extend ordinary holography to its discrete variant!
5. Algebraic variety is rather simple object as compared to the solutions of partial differential equations encountered in physics - say those for minimal surfaces. Now one must fix boundary values or initial values at  $n - 1$ -dimensional surface to fix the solution. For integrable theories the situation can change. In TGD SH suggests that the classical solutions are determined by data at 2-surfaces, which together with conformal invariance could reduce the data to one-dimensional data specified by a polynomial.  $M^8 - H$  correspondence allows to consider this option seriously.
6.  $M^8 - H$  duality suggests that space-time surfaces are co-dimension  $d_c = 4$  algebraic curves in  $M^8$ . Could space-time surfaces define closed sets for the analog of Zariski topology? Could string world sheets and partonic 2-surfaces do the same inside space-time surfaces? An interesting question is whether this generalizes also to the level of imbedding space  $H$  and could perhaps define a topology rougher than real topology in better accord with the notion of finite measurement resolution.

## 2.2 About algebraic curves and surfaces

The realization  $M^8 - H$  correspondence to be considered allows to understand space-time surfaces as 4-D complex algebraic surfaces  $X_c^4$  in the space  $o$  of complexified octonions projected to real sub-space of  $O^c$  with Minkowskian signature. Due to the non-commutativity of quaternions, the reduction of space-time surfaces to curves in quaternionic plane is not possible. Despite this it is instructive to start from the algebraic geometry of curves and surfaces.

### 2.2.1 Degree and genus of the algebraic curve

Algebraic curve is defined as zero locus of a polynomial  $P(x^1, x^2, \dots, x^n)$  with  $x^n$  in some - preferably algebraically closed - number field  $K$  and coefficients in some number field  $k \subset K$ . In adelic physics  $K$  corresponds to real or complex numbers and  $k$  to the extension of rationals defining adèles. In p-adic sectors  $k$  corresponds to the extension of p-adic numbers induced by  $k$ . In general roots belong to an extension of  $k$ .

Degree, genus, and Euler characteristic are the basic characterizers of algebraic curve.

1. The degree  $d$  of algebraic curve corresponds to the highest power for the variables appearing in the polynomial. One can also define multi-degree in an obvious manner. A useful geometric interpretation for the degree is that line intersects curve (also complex) of degree  $d$  in at most  $d$  points as is clear from the fact that the equation of curve reduces the equation for curve to an equation for the roots of  $d$ :th order polynomial of single variable.
2. Also the genus  $g$  of the curve (see <http://tinyurl.com/ybm3wfue>) is important characteristic. One can distinguish between topological genus, geometric genus and arithmetic genus. For curves these notions are equivalent. The connection between genus and degree  $d$  of non-singular algebraic curve is very useful:

$$g = \frac{(d-1)(d-2)}{2} . \quad (2.1)$$

Spherical topology for complex curves corresponds to  $n = 1$  and  $n = 2$ .

A more general formula reads as:

$$g = \frac{(d-1)(d-2)}{2} + \frac{n_s}{2} . \quad (2.2)$$

Here  $n_s$  is the number of holes of the curve behaving like holes and increasing the genus.

3. Euler characteristic (for Euler characteristic see <http://tinyurl.com/pp52zd4>) is a homological invariant making sense in arbitrary dimension and also for manifolds. Homological definition based on simplicial homology relies on counting of simplexes of various dimension. The definition in terms of dimensions of homology groups  $H_n$  is given by

$$\chi = b_0 - b_1 + b_2 \dots + (-1)^n b_n , \quad (2.3)$$

where  $b_k$  is the dimension of  $k$ :th homology group (see <http://tinyurl.com/j48ojys>).

The following gives the engineering rules for obtaining Euler characteristic of the surface obtained from simpler building blocks. Note that algebraic variety property is not essential here.

1. Euler characteristic is homotopy invariant so that it does not change one adds homologically trivial space such as  $E^n$  as a Cartesian factor.
2.  $\chi$  is additive under disjoint union. Inclusion-exclusion principle states that if  $M$  and  $N$  intersect, one has  $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$ .
3. Euler characteristic for the connected sum  $A \# B$  of  $n$ -dimensional manifolds obtained by drilling balls  $B^n$  from summands, giving opposite orientation to the boundaries of the hole, and connecting with cylinder  $D \times S^{n-1}$  is given by  $\chi(A) + \chi(B) - \chi(S^{n-1})$ . One has  $\chi(S^2) = 2$  and  $\chi(D^2) = 1$ .
4. The Euler characteristic for product  $M \times N$  is  $\chi(M) \times \chi(N)$ .
5. The Euler characteristic for  $N$ -fold covering space  $M_n$  is  $N \times \chi(M)$  with a correction term coming from the singularities of the covering (ramified covering space).
6. For a fibration  $M \rightarrow B$  with fiber  $S$ , which differs from fiber bundle in that the fibers are only homeomorphic, one has  $\chi(M) = \chi(B) \times \chi(S)$ .

Euler characteristic and the genus of 2-surface (or complex) curve are related by the equation

$$\chi = 2(1 - g) . \quad (2.4)$$

having values 2, 0, -2, ..... If the 2-surface has  $n_s$  holes (punctures), one has

$$\chi = 2(1 - g) - n_s . \quad (2.5)$$

Punctures must be distinguished from singularities at which some sheets of covering meet at single point.

A formal generalization of the definition of genus for varieties in terms of Euler characteristic makes sense.

$$g = -\frac{\chi}{2} + 1 - \frac{n_s}{2} . \quad (2.6)$$

Disk has genus 1/2 and drilling of  $n$  holes increases genus by  $n/2$ . Pair of holes gives same contribution to  $g$  and the cylinder connecting the holes. Note that for complex curves the definition of puncture is obvious. For real curves the puncture would mean missing point of the curve.

The latter definitions of genus can be identified in terms of Euler characteristic also for higher-dimensional varieties. For curves these notions are equivalent if there are no singularities. For algebraic curves  $g$  is same for the real and complex variants of the curve in  $RP_1$  and  $CP_1$  respectively.

### 2.2.2 Elliptic curves and elliptic surfaces

Elliptic curves (see <http://tinyurl.com/lovksny>) are cubic curves with no singularities (cusps or self-intersections) having representation of form  $y^2 - x^3 - ax - b = 0$ . These singularities can occur only at special values of parameters ( $a = 0, b = 0$ ). Since the degree equals to  $d = 3$ , elliptic curve has genus  $g = 1$ .

Elliptic curves allow a group of Abelian symmetries generated by a finite number of generators. The emergence of abelian group structure can be intuitively understood as follows.

1. Given line intersects the curve of degree 3 in at most 3 points. Let  $P$  and  $Q$  be two of these points. Then there can be also a third intersection point  $R$  and by the  $Z^2$  symmetry changing the sign of  $y$  also the reflection of  $R$  - identify it as  $-R$  - belongs to the curve. Define the sum of  $P + Q$  to be  $-R$ .

The actual proof is slightly more complicated since the number of intersection points for the line with curve can be also 2 or 1. By writing explicit expressions for the coordinates  $x_R$  and  $y_R$ , one can also find that they are indeed rational if the points  $P$  and  $Q$  are rational. If the elliptic curve as single rational point it has infinite number of them.

2. The generators with finite order give rise to torsion. The rank of generators of infinite order is called rank and conjectured to be arbitrarily large (see <http://tinyurl.com/lovksny>). Therefore elliptic curve is an Abelian group and one talks about Abelian variety. If elliptic curve contains a rational point it contains entire lattice of rational points obtained as shifts of this point.

**Remark:** Complex elliptic curves are 2-surfaces in complex projective plane  $CP_2$  and therefore highly interesting from TGD point of view.  $g = 1$  partonic 2-surfaces would in TGD framework correspond to second generation fermions [K1]. Abelian varieties define a generalization of elliptic curves to higher dimensions and simplest space-time surfaces allowing also large cognitive representations could correspond to such.

Elliptic surfaces (see <http://tinyurl.com/yc33a6dg>) are fibrations with an algebraic curve as base space and elliptic curve as fiber (fibration is more general notion than fiber space since the fibers are only homeomorphic). The singular fibers failing to be elliptic curves have been classified by Kodaira.

## 2.3 The notion of rational point and its generalization

The notion of algebraic integer (see <http://tinyurl.com/y8z389a7>) makes sense for any number field as a root of a monic polynomial (polynomial with integer coefficients with coefficient of highest power equal to unity). The field of fractions for given number field consists of ratios of algebraic integers. The same is true for the notion of prime. The more precise definition forces to replace integers and primes with ideals.

Rational varieties are expressible as maps defined by rational functions with rational coefficients in some extension of  $Q$  and contain infinite number of rational points. If the variety is not rational, one can ask whether it could allow a dense set of rational points with rational number replaced with the ratio of algebraic integers for some extension of  $Q$ . This leads to the idea of potentially rational point, and one can classify algebraic varieties according to whether they are potentially rational or not. The variety is potentially rational if it allows a parametric representation using rational functions. Otherwise the parametric representation involves algebraic functions such as roots of rational functions.

The interpretation in terms of cognition would be that large enough extension makes the situation “cognitively easy” since cognitive representations involving fermions at the rational points and defining discretizations of the algebraic variety could be arbitrary large. The simpler the surface is cognitively, the larger the number of rational points or potentially rational points is.

Complexity of algebraic varieties is measured by Kodaira dimension  $d_K$  (see <http://tinyurl.com/yadoj2ut>). The spectrum for this dimension varies in the range  $(-\infty, 0, 1, 2, \dots, d)$ , where  $d$  is the algebraic dimension of the variety. Maximal value equals to the ordinary topological dimension  $d$  and corresponds to maximal complexity: in this case the set of rational points is finite. Minimal Kodaira dimension is  $d_K = -\infty$ : in this case the set of rational points is infinite. Rational surfaces

are maximally simple and this corresponds to the existence of parametric representations using only rational functions.

### 2.3.1 Rational points for algebraic curves

The sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see <http://tinyurl.com/y9oq37ce>) states that a curve over  $Q$  with genus  $g = (d-1)(d-2)/2 > 1$  (degree  $d > 3$ ) has only finitely many rational points.

1. Sphere  $CP_1$  in  $CP_2$  has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of  $SU(2)$ ) allow dense set of rational points [A3, A5].

$g = 0$  does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in  $CP_2$  with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve  $y^2 - x^3 - ax - b$  in  $CP_2$  (see <http://tinyurl.com/lovksny>) has genus  $g = 1$  and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for  $a = 0, b = 0$  origin is a singularity).

$g = 1$  does not guarantee that there is infinite number of rational points. Fermat's last theorem and  $CP_2$  as example.  $x^d + y^d = z^d$  is projectively invariant statement and therefore defines a curve with genus  $g = (d-1)(d-2)/2$  in  $CP_2$  (one has  $g = 0, 0, 2, 3, 6, 10, \dots$ ). For  $d > 2$ , in particular  $d = 3$ , there are no rational points.

3.  $g \geq 2$  curves do not allow a dense set of rational points nor even potentially dense set of rational points.

**Remark:** In TGD framework algebraic varieties could be zero loci of octonionic polynomials and have algebraic dimension 4 so that the classification for algebraic curves does not help. Octonion analyticity must bring in symmetries which simplify the situation.

### 2.3.2 Enriques-Kodaira classification

The tables of (see <http://tinyurl.com/ydelr4np>) give an overall view about the Enriques-Kodaira classification of algebraic curves, surfaces, and varieties in terms of Kodaira dimension (see <http://tinyurl.com/yadoj2ut>).

1. For instance, general curves ( $g \geq 2$ ) have  $d_K = 1$ , elliptic curves ( $g = 1$ ) have  $d_K = 0$  and projective line ( $g = 0$ ) has  $d_K = -\infty$ .  $CP_1 \subset CP_2$  is a rational curve so that rational points are dense. Elliptic curves allow infinite number or rational points forming an Abelian group if they containing single rational point and are therefore cognitively easy.
2. Algebraic varieties (with real dimension  $d_R = 4$  in complex case) with  $d_K = 2$  are surfaces of general type, elliptic surfaces (see <http://tinyurl.com/yc33a6dg>) have  $d_K = 1$ , surfaces with attribute abelian, hyper-elliptic, K3, and Enriques, have  $d_K = 0$ .

**Remark:** All real 2-surfaces are hyper-elliptic for  $g \leq 2$ , in other words allow  $Z_2$  as global conformal symmetry. Genus-generation correspondence [K1] for fermions allows to assign to the 3 lowest generations of fermions hyper-elliptic partonic 2-surfaces with genus  $g = 0, 1, 2$ . These surfaces would have  $d_K = 0$  and be rather simple as real surfaces in Kodaira classification. Could one regard them as  $M^4$  projection of complex hyper-elliptic surfaces of real dimension  $d_R = 4$ ?  $d_K = -\infty$  holds true for rational surfaces and ruled surfaces, which allow straight line through any point are maximally simple. In complex case the line would be  $CP_1$ .

3. The Wikipedia article gives also a table about the classification of algebraic 3-folds. Real algebraic 3-surfaces might well occur in TGD framework. The twistor space for space-time surface might allow realization as complex 3-fold and since it has  $S^2$  as fiber, it would naturally correspond to a uni-ruled surface with  $d_K = -\infty$ . The table shows that one can build higher dimensional algebraic varieties with  $d_K < d$  from lower-dimensional ones as fiber-space like structures, which based on fiber having  $d_K < d$ . 3-D Abelian varieties and Calabi-Yau 3-folds are complex manifolds with  $d_K = 0$ , which cannot be engineered in this manner.
4. Space-time surfaces would be surfaces of algebraic dimension 4. Wikipedia tables do not give direct information about this case but one can make guesses on basis of the three tables. Octonionic polynomials are analytic continuations of real polynomials of real variable, which must mean a huge simplification, which also favor cognitive representability. The best that one might have infinite sets of rational points. The examples about extremals of Kähler action does not however favor this wish.

Bombieri-Lang conjecture (see <http://tinyurl.com/y887yn5b>) states that, for any variety  $X$  of general type over a number field  $k$ , the set of  $k$ -rational points of  $X$  fails to be Zariski dense (see <http://tinyurl.com/jm9fh74>) in  $X$ . This means that, the  $k$ -rational points are contained in a *finite* union of lower-dimensional sub-varieties of  $X$ . In dimension 1, this is exactly Faltings theorem, since a curve is of general type if and only if it has  $g \geq 2$ . The conjecture of Vojta (see <http://tinyurl.com/y9sttuu4>) states that varieties of general type cannot be potentially dense. As will be found, these conjectures might be highly relevant for TGD.

### 3 About enumerative algebraic geometry

Algebraic geometry is something very different from Riemann geometry, Kähler geometry, or sub-manifold geometry based on local notions. Sub-manifolds are replaced with sub-varieties defined as zero loci for polynomials with coefficients in the field of rationals or extension of rationals. Partial differential equations are replaced with algebraic ones. One can generalize algebraic geometry to any number field.

String theorists have worked with algebraic geometry with motivation coming from various moduli spaces emerging in string theory. The moduli spaces for closed and open strings possibly in presence of branes are involved. Also Calabi-Yau compactification leads to algebraic geometry, and topological string theories of type A and B involve also moduli spaces and enumerative algebraic geometry.

In TGD the motivation for enumerative algebraic geometry comes from several sources.

1. Twistor lift of TGD lifts space-time surfaces to their 6-D twistor spaces representable as surfaces in the product of 6-D twistor spaces of  $M^4$  and  $CP_2$  and possessing Kähler structure - this makes these spaces completely unique and strongly suggests the role of algebraic geometry, in particular in the generalization of twistor Grassmannian approach [L5].
2. There are three threads in number theoretic vision: p-adic numbers and adelic, classical number fields, and infinite primes. Adelic physics [L7] as physics of sensory experience and cognition unifies real physics and various p-adic physics in the adèle characterized by an extension of rationals inducing those of p-adic number fields. This leads to algebraic geometry and counting of points with imbedding space coordinates in the extension of rationals and defining a discrete cognitive representation. The core of the scattering amplitude would be defined by this cognitive representation identifiable in terms of points appearing as arguments of n-point function in QFT picture [L4].
3.  $M^8 - M^4 \times CP_2$  duality is the analog of the rather adhoc spontaneous compactification in string models but would be non-dynamical and thus allow to avoid landscape catastrophe. Classical physics would reduce to octonionic algebraic geometry at the level of complexified octonions with several special features due to non-commutativity and non-associativity: space-time could be seen as 4-surface in the complexification of octonions. The commuting imaginary unit would make possible the realization of algebraic extensions of rationals.



The moduli space for the varieties is discrete if the coefficients of the polynomials are in the extension of rationals. If one poses additional conditions such as associativity of 4-surfaces, the moduli space is further reduced by the resulting criticality conditions realizing quantum criticality at the fundamental level raising hopes about extremely simple formulation of scattering amplitudes at the level of  $M^8$  [L5].

Also complex and co-complex sub-manifolds of associative space-time surface are important and would realize strong form of holography (SH). For non-associative regions of space-time surface it might not be possible to define complex and co-complex surfaces in unique manner since the basic  $M^2 \subset M^4$  local flag structure is missing. String world sheets and partonic 2-surfaces and their moduli spaces are indeed in key role and the topology of partonic surfaces plays a key role in understanding of family replication phenomenon in TGD [L4].

In this framework one cannot avoid enumerative algebraic geometry.

1. One might want to know the number of points of sub-variety belonging to the number field defining the coefficients of the polynomials. This problem is very relevant in  $M^8$  formulation of TGD, where these points are carriers of sparticles. In TGD based vision about cognition [L7] they define cognitive representations as points of space-time surface, whose  $M^8$  coordinates can be thought of as belonging to both real number field and to extensions of various p-adic number fields induced by the extension of rationals. If these cognitive representations define the vertices of analogs of twistor Grassmann diagrams in which sparticle lines meet, one would have number theoretically universal adelic formulation of scattering amplitudes and a deep connection between fundamental physics and cognition.
2. Second kind of problem involves a set algebraic surfaces represented as zero loci for polynomials - lines and circles in the simplest situations. One must find the number of algebraic surfaces intersecting or touching the surfaces in this set. Here the notion of incidence is central. Point can be incident on line or two lines (being their intersection), line on plane, etc.. This leads to the notion of Grassmannians and flag-manifolds.

Moduli spaces parameterizing sub-varieties of given kind - lines, circles, algebraic curves of given degree, are central for the more advanced formulation of algebraic geometry. These moduli spaces emerge also in the formulation of TGD. The moduli space of conformal equivalence classes of partonic 2-surfaces is one example involved with the explanation of family replication phenomenon [K1]. One can assign moduli spaces also to octonion and quaternion structures in  $M^8$  (or equivalently with the complexification of  $E^8$ ). One can identify  $CP_2$  as a moduli space of quaternionic sub-spaces of octonions containing preferred complex sub-space.

One cannot avoid these moduli spaces in the formulation of the scattering amplitudes and this leads to  $M^8 - H$  duality. The hard core of the calculation should however reduce to the understanding of the algebraic geometry of 4-surfaces in octonionic space. Clearly,  $M^8$  picture seems to provide the simplest formulation of the number theoretic vision.

### 3.1 Some examples about enumerative algebraic geometry

Some examples give an idea about what enumerative algebraic geometry (see <http://tinyurl.com/y7yzt67b>) is.

1. Consider 4 lines in 3-D space. What is the number of lines intersecting these 4 lines [A8] (see <http://tinyurl.com/ycrbr5aj>). One could deduce the number of lines and lines by writing the explicit equations for the lines with each line characterized by  $2+3=5$  parameters specifying direction  $t$  vector and arbitrarily chosen point  $x_0$  on the line.  $2+3=5$  parameters characterize each sought-for line.

For intersection points  $x_i$  of sought for line with  $i$ :th one has  $x_i = x_0 + k_i t_0$ ,  $i = 1, \dots, 4$  for the sought for line with direction  $t_0$ . At the intersection points at the 4 lines one has  $x_i = x_{0i} + K_i t_i$  with fixed directions  $t_i$ . Combining the two equations for each line one has  $4 \times 3 = 12$  equations and  $3+4+2$  parameters for the sought for line plus 4 parameters  $K_i$  for the four lines. This gives 13 unknown parameters corresponding to  $x_0, k_i, K_i$ . One would have one parameter set of solutions: something goes wrong.

One has however projective invariance: one can shift  $x_0$  along the line by  $x_0 \rightarrow x_0 - at$ ,  $k_i \rightarrow k_i + a$  and using this freedom assume for instance  $k_1 = 0$ . This reduces the number of parameters to 12 and one has finite number of solutions in the generic case. Actually the number is 2 in the generic case but can be infinite in some special cases. The challenge is to deduce the number of the solutions by geometric arguments. Below Schubert's argument proving that the number of solutions is 2 will be discussed.

The idea of enumerative geometry is to do this using general geometric arguments allowing to deform the problem topologically to a simpler one in which case the number of solutions is obvious which in the most abstract formulation become topological.

2. Apollonius can be seen as founder of enumerative algebraic geometry. Apollonian circles (see <http://tinyurl.com/ycvxe688>) represent second example. One has 3 circles in plane. What is the number of circles tangential to all these 3 circles. Wikipedia link represents the geometric solution of the problem. The number of circles is 8 in the generic case but there are exceptional cases.
3. In Steiner's conic problem (see <http://tinyurl.com/yahshsjo>) one have 5 conical sections (circles, cones, ellipsoids, hyperbole) in plane. How many different conics tangential to the conics there exist? This problem is rather difficult and the thumb rules of enumerative geometry (dimension counting, Bezout's rule, Schubert calculus) fail. This is a problem in projective geometry where one is forced to introduce moduli space for conics tangential to given conic. This space is algebraic sub-variety of all conics in plane which is 5-D projective space. One must be able to deduce the number of points in the intersection of these sub-varieties so that the original problem in 2-D plane is replaced with a problem in moduli space.

### 3.2 About methods of algebraic enumerative geometry

A brief summary about methods of algebraic geometry is in order to give some idea about what is involved (see <http://tinyurl.com/y7yzt67b>).

1. Dimension counting is the simplest method. If two geometric objects of  $n$ -D space have dimensions  $k$  and  $l$ , there intersection is  $n - k - l$ -dimensional for  $n - k - l \geq 0$  or empty in the generic case. For  $k + l = n$  one obtains discrete set of intersection points.
2. Bezouts theorem is a more advanced method. Consider for instance, curves in plane defined by the curves polynomials  $x = P^m(y)$  and  $x = P^n(y)$  of degrees  $k = m$  and  $k = n$ . The number  $N$  of intersection points in the generic case is bounded above by  $N = m \times n$  (in this case all roots are real). One can understand this by noticing that one has  $m$  roots  $y_k$  or given  $x$  giving rise to a  $m$ -branched graph of function  $y = f(x)$ . The number of intersections for the graphs of the two polynomials is at most  $m \times n$ . If one has curve in plane represented by polynomial equation  $P^{m,n}(x, y) = 0$ , one can also estimate immediately the minimal multi-degree  $(m, n)$  for this polynomials.
3. Schubert calculus <http://tinyurl.com/y766ddw2>) is a more advanced but not completely rigorous method of enumerative geometry [A8] (see <http://tinyurl.com/ycrbr5aj>).

Schubert's vision was that the number of intersection points is stable against deformations in the generic case. This is not quite true always but in exceptional cases one can say that two separate solutions degenerate to single one, just like roots of polynomial can do for suitable values of coefficients.

For instance, Schubert's solution to the already mentioned problem of finding a line intersecting 4 lines in generic position relies on this assumption. The idea is to deform the situation so that one has two intersecting pairs of lines. One solution to the problem is a line going through the intersection points for line pairs. Second solution is obtained as intersection of the planes. It can happen that planes are parallel in which case this does not work.

Schubert calculus it applies to linear sub-varieties but can be generalized also to non-linear varieties. The notion of incidence allowing a general formulation for intersection and tangentiality (touching) is central. This leads to the notions of flag, flag manifold, and Schubert variety as sub-variety of Grassmannian.

Flag is a hierarchy of incident subspaces  $A_0 \subset A_1 \subset A_2 \dots \subset A_n$  with the property that the dimension  $d_i \leq n$  of  $A_i$  satisfies  $d_i \geq i$ . As a special case this notion leads to the notion of Grassmannian  $G(k, n)$  consisting of  $k$ -planes in  $n$ -dimensional space: in this case  $A_0$  corresponds to  $k$ -planes and  $A_2$  to space  $A_n$ . More general flag manifolds are moduli spaces and sub-varieties of Grassmannian providing a solution to some conditions. Flag varieties as sub-varieties of Grassmannians are Schubert varieties (see <http://tinyurl.com/y7ehcrzg>). They are also examples of singular varieties. More general Grassmannians are obtained as coset spaces of  $G/P$ , where  $G$  is algebraic group and  $P$  is parabolic sub-group of  $G$ .

**Remark:**  $CP_2$  corresponds to the space of complex lines in  $C^3$ .  $CP_2$  can be also understood as the space of quaternionic planes in octonionic 8-space containing fixed 2-plane so that also now one has flag. String world sheets inside space-time surfaces define curved flags with 2-D and 4-D tangent spaces defining an integrable distribution of local flags.

4. Cohomology combined with Poincare duality allows a rigorous formulation of Schubert calculus. Schubert's idea about possibility to deform the generic position corresponds to homotopy invariance, when the degeneracies of the solutions are taken into account. Homology and cohomology become basic tools and the so called cup product for cohomology together with Poincare duality and Künneth formula for the cohomology of Cartesian product in terms of cohomologies of factors allows to deduce intersection numbers algebraically. Schubert cells define a basis for the homology of Grassmannian containing only even-dimensional generators.

Grassmannians play a key role in twistor Grassmannian approach as auxiliary manifolds. In particular, the singularities of the integrand of the scattering amplitude defined as a multiple residue integral over  $G(k, n)$  define a hierarchy of Schubert cells. The so called positive Grassmannian [B2] defines a subset of singularities appearing in the scattering amplitudes of  $\mathcal{N} = 4$  SUSY. This hierarchy and its  $CP_2$  counterpart are expected also in TGD framework.

**Remark:** Schubert's vision might be relevant for the notion of conscious intelligence. Could problem solving involve the transformation of a problem to a simple critical problem, which is easy but for which some solutions can become degenerate? The transformation of general position for 4 lines to a pair of intersecting lines would be example of this. One can wonder whether quantum criticality could help problem solving by finding critical cases.

5. Moduli spaces of curves and varieties provide the most refined methods. Flag manifolds define basic examples of moduli spaces. Quantum cohomology represents even more refined conceptualization: the varieties (branes in M-theory terminology) are said to be connected or intersect if each of them has a common point with the same pseudo-holomorphic variety ("string world sheet"). Pseudo-holomorphy - which could have minimal surface property as counterpart - implies that the connecting 2-surface is not arbitrary.

Quantum intersection for the "string world sheet" and "brane" is possible also when it is not stable classically (the co-dimension of brane is smaller than 2). Even in the case that it possible classically quantum intersection makes possible kind of "telepathic" quantum contact mediated by the "string world sheet" naturally involved with the description of quantum entanglement in TGD framework.

### 3.3 Gromow-Witten invariants

Gromow-Witten invariants represent example of so called quantum invariants natural in string models and M-theory. They provide new invariants in algebraic and symplectic geometry.

#### 3.3.1 Formal definition

Consider first the definition of Gromow-Witten (G-W) invariants (see <http://tinyurl.com/y9b5vbcw>). G-W invariants are rational number valued topological invariants useful in algebraic

and symplectic geometry. These quantum invariants give information about these geometries not provided by classical invariants. Despite being rational numbers in the general case G-W invariants in some sense give the number of string world sheets connecting given branes.

1. One considers collection of  $n$  surfaces (“branes”) with even dimensions in some symplectic manifold  $X$  of dimension  $D = 2k$  (say Kähler manifold) and pseudo-holomorphic curves (“string world sheets”)  $X^2$ , which have the property that they connect these  $n$  surfaces in the sense that they intersect the “branes” in the marked points  $x_i, i = 1, \dots, n$ .

“Connect” does not reduce to intersection in topologically stable sense since connecting is possible also for branes with dimension smaller than  $D - 2$ . One allows all surfaces that  $X^2$  that intersects the  $n$  surfaces at marked points if they are pseudo-holomorphic even if the basic dimension rule is not satisfied. In 4-dimensional case this does not seem to have implications since partonic 2-surfaces satisfy automatically the dimension rule. The  $n$  branes intersect or touch in quantum sense: there is no concrete intersection but intersection with the mediation of “string world sheet”.

2. Pseudo-holomorphy means that the Jacobian  $df$  of the imbedding map  $f : X^2 \rightarrow X$  commutes with the symplectic structures  $j$  resp.  $J$  of  $X^2$  resp.  $X$ : i.e. one has  $df(jT) = Jdf(T)$  for any tangent vector  $T$  at given point of  $X^2$ . For  $X^2 = X = C$  this gives Cauchy-Riemann conditions.

In the symplectic case  $X^2$  is characterized topologically by its genus  $g$  and homology class  $A$  as surface of  $X$ . In algebraic geometry context the degree  $d$  of the polynomial defining  $X^2$  replaces  $A$ . In TGD  $X^2$  corresponds to string world sheet having also boundary.  $X^2$  has also  $n$  marked points  $x_1, \dots, x_n$  corresponding to intersections with the  $n$  surfaces.

3. G-W invariant  $GW_{g,n}^{X,A}$  gives the number of pseudo-holomorphic 2-surfaces  $X^2$  connecting  $n$  given surfaces in  $X$  - each at single marked point. In TGD these surfaces would be partonic 2-surfaces and marked points would be carriers of sparticles.

The explicit definition of G-W invariant is rather hard to understand by a layman like me. I however try to express the basic idea on basis of Wikipedia definition (see <http://tinyurl.com/y9b5vbcw>). I apologize for my primitive understanding of higher algebraic geometry. The article of Vakil [A6] (see <http://tinyurl.com/ybobccub>) discusses the notion of G-W invariant in detail.

1. The situation is conformally invariant meaning that one considers only the conformal equivalence classes for the marked pseudo-holomorphic curves  $X^2$  parameterized by the points of so called Deligne-Mumford moduli space  $\overline{M}_{g,n}$  of curves of genus  $g$  with  $n$  marked points (see <http://tinyurl.com/yaq8n6dp>): note that these curves are just abstract objects without no imbedding as surface to  $X$  assumed.  $\overline{M}_{g,n}$  has complex dimension

$$d_0 = 3(g - 1) + n .$$

$n$  corresponds complex dimensions assignable to the marked points and  $3(g - 1)$  correspond to the complex moduli in absence of marked points. This space appears in TGD framework in the construction of elementary particle vacuum functionals [K1].

2. Since these curves must be represented as surfaces in  $X$  one must introduces the moduli space  $\overline{M}_{g,n}(X, A)$  of their maps  $f$  to  $X$  with given homology equivalence class. The elements in this space are of form  $(C, x_1, \dots, x_n, f)$  where  $C$  is one particular representative of  $A$ .
3. The complex dimension  $d$  of  $\overline{M}_{g,n}(X, A)$  can be calculated. One has

$$d = d_0 + c_1^X(A) + (g - 1)k .$$

Here  $c_1^X(A)$  is the first Chern class defining element of second cohomology of  $X$  evaluated for  $A$ . For Calabi-Yau manifolds one has  $c_1 = 0$ . The contribution  $(g - 1)k$  to the dimension vanishing for torus topology should have some simple explanation.

4. One defines so called evaluation map  $ev$  from  $\overline{M}_{g,n}(X, A) \rightarrow Y$ ,  $Y = \overline{M}_{g,n} \times X^n$  in terms of stabilization  $st(C, x_1, \dots, x_n) \in \overline{M}_{g,n}(X, A)$  of  $C$  (I understand that stabilization means that the automorphism group of the stabilized surface defined by  $f$  is finite [A7] (see <http://tinyurl.com/y8r44uhl>). I am not quite sure what the finiteness of the automorphism group means. One might however think that conformal transformations must be in question. One has

$$ev(C, x_1, \dots, x_n, f) = (st(C, x_1, \dots, x_n), f(x_1), \dots, f(x_n)) .$$

Evaluation map assigns to the concrete realization of string world sheet with marked points the abstract curve  $st(C, x_1, \dots, x_n)$  and points  $(f(x_1), \dots, f(x_n)) \in X^n$  possibly interpretable as positions  $f(x_i)$  of  $n$  particles. One could say that one has many particle system with particles represented by surfaces of  $X_i$  of  $X$  connected by  $X^2$  - string world sheet - mediating interaction between  $X_i$  via the intersection points.

5. Evaluation map takes the fundamental class of  $\overline{M}_{g,n}(X, A)$  in  $H_d(\overline{M}_{g,n}(X, A))$  to an element of homology group  $H_d(Y)$ . This homology equivalence class defines G-W invariant, which is rational valued in the general case.
6. One can make this more concrete by considering homology equivalence class  $\beta$  in  $\overline{M}_{g,n}$  and homology equivalence classes  $\alpha_i$ ,  $i = 1, \dots, n$  represented by the surfaces  $X_i$ . The co-dimensions of these  $n+1$  homology equivalence classes must sum up to  $d$ . The homologies of  $\overline{M}_{g,n}$  and  $Y = \overline{M}_{g,n} \times X^n$  induce homology of  $Y$  by Künneth formula (see <http://tinyurl.com/yd9tt1fr>) implying that  $Y$  has class of  $H_d(Y)$  given by the product  $\beta \cdot \alpha_1 \dots \alpha_n$ .

One can identify the value of  $GW_{g,n}^{X,A}$  for a given class  $\beta \cdot \alpha_1 \dots \alpha_n$  as the coefficients in its expansion as sum of all elements in  $H_d(Y)$ . This coefficient is the value of its intersection product of  $GW_{g,n}^{X,A}$  with the product  $\beta \cdot \alpha_1 \dots \alpha_n$  and gives element of  $H_0(Q)$ , which is rational number.

7. There are two non-classical features. Classically intersection must be topologically stable. This would require  $\alpha_i$  to have codimension 2 but all even co-dimensions are allowed. That the value for the number of connecting string world sheets is rational number does not conform with the classical geometric intuition. The Wikipedia explanation is that the orbifold singularities for the space  $\overline{M}_{g,n}(X, A)$  of stable maps are responsible for rational number.

### 3.3.2 Application to string theory

Topological string theories give a physical realization of this picture. Here the review article *Instantons, Topological Strings, and Enumerative Geometry* of Szabo [A7] (see <http://tinyurl.com/y8r44uhl>) is very helpful.

1. In M-theory framework and for topological string models of type A and B the physical interpretation for the varieties associated with  $\alpha_i$  would be as branes of various dimensions needed to satisfy Dirichlet boundary conditions for strings.
2. In topological string theories one considers sigma model with target space  $X$ , which can be rather general. The symplectic or complex structure of  $X$  is however essential.  $X$  is forced to be 3-D (in complex sense) Calabi-Yau manifold by consistency of quantum theory. Interestingly, the super twistor space  $CP(3|4)$  is super Calabi-Yau manifold although  $CP_3$  is not and must therefore have trivial first Chern class  $c_1$  appearing in the formula for the dimension  $d$  above. I must admit that I do not understand why this is the case.

Closed topological strings have no marked points and one has  $n = 0$ . Open topological strings world sheets meet  $n$  branes at points  $x_i$ , where they satisfy Dirichlet boundary conditions. Branes can be identified as even-dimensional Lagrangian sub-manifolds with vanishing induced symplectic form.

3. For topological closed string theories of type A one considers holomorphically imbedded curves in  $X$  characterized by genus  $g$  and homology class  $A$ : one speaks of world sheet

instantons.  $A = \sum n_i S_i$  is sum over the generating classes  $S_i$  with integer coefficients. For given  $g$  and  $A$  one has analog of product of non-interacting systems at temperatures  $1/t_i$  assignable to the homology classes  $S_i$  with energies identifiable as  $n_i$ . One can assign Boltzmann weight labelled by  $(g, A)$  as  $Q^\beta = \prod_i Q_i^{n_i}$ ,  $Q_i = \exp(-t_i)$ .

One can construct partition function for the entire system as sum over Boltzmann weights with degeneracy factors telling the number of world sheet instantons with given  $(g, A)$ . One can calculate free energy as sum  $\sum N_{g,\beta} Q^\beta$  over contributions labelled by  $(g, A)$ . The coefficients  $N_{g,\beta}$  count the rational valued degeneracies of the world sheet instantons of given type and reduce to G-W invariants  $GW_{g,0}^{X,A}$ .

**Remark:** If one allows powers of a root  $e^{-1/n}$ ,  $t = n$ , in the extension of rationals or replace  $e^{-t}$  with  $p^n$ , partition functions make sense also in the p-adic context.

4. For topological open string theories of type A one has also branes. Homology equivalence classes are relative to the brane configuration. The coefficients  $N_{g,\beta}$  are given by  $GW_{g,n}^{X,A}$  for a given configuration of branes: the above described general formulas correspond to these.
5. For topological string theories of type B, string world sheets reduce to single point and thus correspond to constant solutions to the field equations of sigma model. Quantum intersection reduces to ordinary intersection and one has  $x_1 = x_2 \dots = x_n$ . G-W invariants involve only classical cohomology and give for  $n = 2$  the number of common points for two surfaces in  $X$  with dimension  $d_1$  and  $d_2 = n - d$ . The duality between topological string theories of type A and B related to the mirror symmetry supports the idea that one could generalize the calculation of these invariants in theories B to theories A. It is not clear whether this option as any analog in TGD.

The so called Witten conjecture (see <http://tinyurl.com/yccahv3q>) proved by Kontsevich states that the partition function in one formulation of stringy quantum gravity and having as coefficients of free energy G-W invariants of the target space is same as the partition function in second formulation and expressible in terms of so called tau function associated with KdV hierarchy. This leads to non-trivial identities. Witten conjecture actually follows from the invariance of partition function with respect to half Virasoro algebra and Virasoro conjecture (see <http://tinyurl.com/y7xcc9hm>) stating just this generalizes Witten's conjecture.

### 3.4 Riemann-Roch theorem

Riemann-Roch theorem (RR) is also part of enumerative geometry albeit more abstract. Instead of counting of numbers of points, one counts dimensions of various function spaces associated with Riemann surfaces. RR provides information about the dimensions for the spaces of meromorphic functions and 1-forms with prescribed zeros and poles.

#### 3.4.1 Basic notions

Riemann surface is the basic notion. Riemann surface is orientable is characterized by its genus  $g$  and number of holes/punctures in it. Riemann surface can also possess marked points, which seem to be equivalent with punctures. The moduli space of these complex curves is parameterized by a moduli space  $\overline{M}_{g,n}$  of curves of genus  $g$  with  $n$  marked points (see <http://tinyurl.com/yaq8n6dp>) (see <http://tinyurl.com/yaq8n6dp>).

Dolbeault cohomology (see <http://tinyurl.com/y7cvs5sx>) generalizes the notion of differential form so that it has well-defined degrees with respect to complex coordinates and their conjugates: one can write in general complex manifold this kind of form as

$$\omega = \omega_{i_1 i_2 \dots i_n, j_1 j_2 \dots j_n} dz^{i_1} \wedge dz^{i_2} \dots dz^{i_n} d\bar{z}^{j_1} \wedge d\bar{z}^{j_2} \dots d\bar{z}^{j_n} .$$

The ordinary exterior derivative  $d$  is replaced with its holomorphic counterpart  $\partial$  and its conjugate. One can construct the counterparts of cohomology groups (Hodge theory)  $H^{p,q} = H^{q,p}$ . Betti numbers as numbers  $h_{i,j}$  defining the dimensions of the cohomology groups forms of degrees  $i$  and  $j$  with respect to  $dz^i$  and  $d\bar{z}^j$ . One can define the holomorphic Euler's characteristic as  $\chi_C = h_{0,0} - h_{0,1} = 1 - g$  whereas ordinary Euler characteristic is  $\chi_R = h_{0,0} - (h_{0,1} + h_{1,0}) + h_{1,1} = 2(1 - g)$ .

One considers meromorphic functions having poles and zeros as the only singularities (points at which the map does not preserve angles): rational functions provide the basic example. Riemann zeta provides example of meromorphic function not reducing to rational function. Holomorphic functions have only zeros and entire functions have neither zeros nor poles. If analytic functions on Riemann surfaces can be interpreted as maps of compact Riemann surface to itself rather than to complex plane, meromorphy reduces to holomorphy since the point  $\infty$  belongs to the Riemann surface.

The elements of free group of divisors are defined as formal sums of integers associated with the points  $P$  of Riemann surface. Divisors  $D = \sum_P n(P)$ , where  $(P)$  is integer, are analogous to integer valued “wave functions” on Riemann surface. The number of points with  $n(P) \neq 0$  is countable. The degree of divisor is obtained as the ordinary sum  $\text{deg}(D)$  of the integers defining the divisor.

Although divisors can be seen as purely formal objects, they are in practice associated to both meromorphic functions and 1-forms. The divisor of a meromorphic function is known as principal divisor. Meromorphic functions and 1-forms differing by a multiplication with meromorphic function are regarded as linearly equivalent - in other words, one can add to a given divisor a divisor of a meromorphic function without changing its equivalence class. It can be shown that all divisors associated with meromorphic 1-forms linearly equivalent and one can talk about canonical divisor. Note that  $\text{deg}(D)$  is linear invariant since the degree of globally meromorphic function is zero.

The motivation for the divisors is following. Consider the space of meromorphic functions  $h$  with the property that the degrees of poles associated with the poles of these functions are not higher than given integers  $n(P)$ . In other words, one has  $\langle h(P) \rangle + D(P) \geq 0$  for all points  $P$  ( $\langle h \rangle$  is the divisor of  $h$ ). For  $D(P) > 0$  the pole has degree not higher than  $D(P)$ . For non-positive  $D(P)$  the function has zero of order  $D(P)$  at least.

### 3.4.2 Formulation of RR theorem

With these prerequisites it is possible to formulate RR (for Wikipedia article see <http://tinyurl.com/mdmbcx6>). The Wikipedia article is somewhat confusing and a more precise description of RR can be found in the article “Riemann-Roch theorem” by Vera Talovikova [A9] (see <http://tinyurl.com/ktww7ks>).

Let  $l(D)$  be the dimension of the space of meromorphic functions with principal divisor  $D$  or 1-forms linearly equivalent with canonical divisor  $K$ . Then the equality

$$l(D) - l(K - D) = \text{deg}(D) - g + 1 \quad (3.1)$$

is true for both meromorphic functions and canonical divisors. For  $D = K$  one obtains using  $l(0) = 1$

$$l(K) = \text{deg}(K) - g + 2 \quad (3.2)$$

giving the dimension of the space of canonical divisors.  $l(K) > 0$  in general is not in conflict with the fact that canonical divisors are linearly equivalent.  $\text{deg}(K) = 2g - 2$  in the above formula gives  $l(K) = g$ .

$l(K) = 0$  for  $g = 0$  case looks strange: one should actually make notational distinction between dimensions of spaces of meromorphic functions and one-forms (this is done in the article of Talovikova). The explanation is that  $l(K)$  here is not the dimension of the space of canonical 1-forms but that of the holomorphic functions with the divisor of  $K$ . The canonical form  $K$  for the sphere has second order pole at  $\infty$  so that one cannot have meromorphic forms holomorphic outside  $P$ .

Riemann’s inequality

$$l(D) \geq \text{deg}(D) - g + 1 \quad (3.3)$$

follows from  $l(K - D) \geq 0$ , which can be seen as a correction term to the formula

$$l(D) = \text{deg}(D) - g + 1 \quad (3.4)$$

In what sense this is true, becomes clear from what follows.

### 3.4.3 The dimension of the space meromorphic functions corresponding to given divisor

The simplest divisor associated with meromorphic function involves only one point. Multiplying a function, which is non-vanishing and finite at  $P$  by  $(z - z(P))^{-n}$  gives a pole of order  $n$  (zero has negative order in this sense). One is interested on the dimension  $l(nP)$  of the space  $nP$  of meromorphic functions and RR allows to deduce information about  $l(nP)$ . One is interested on the behavior of  $l(nP)$  as function of genus  $g$  of Riemann surface (more general situation would allow also punctures). For  $n = 0$  one has entire function without poles and zeros. Only constant function is possible:  $l(0) = 1$ .

First some general observations.  $K$  has degree  $\deg(K) = 2g - 2$ , which gives  $l(K) = g$ . For  $n = \deg(D) > \deg(K) = 2g - 2$  the correction term vanishes since  $\deg(K - D)$  becomes negative, and one has  $l(D) = \deg(D) - g + 1$ . This gives  $l(n = 2g - 1) = g$ . Therefore  $n \in \{2g - 1, 2g, \dots\}$  corresponds to  $l(nP) \in \{g, g + 1, \dots\}$ .  $n < 2g - 2$  corresponds to  $l(nP) = 1$ . What about the range  $n \in \{2, \dots, 2g - 2\}$ ? Note that  $2g - 2$  is the negative of the Euler character of Riemann surface.

1.  $g = 0$  case.  $K$  on sphere.  $dz$  canonical 1-form on Riemann sphere covered by two complex coordinate patches.  $z \rightarrow w = 1/z$  relates the coordinates. There is second order pole at infinity ( $dw = -dz/z^2$ ). One has therefore  $\deg(K) = -2$  for sphere in accordance with the general formula  $\deg(K) = 2g - 2$ . The formula  $l(nP) = \deg(D) + 1$  holds for all  $n$  and there is no correction term now. One as  $l(nP) = n + 1$ .

2.  $g = 1$  case.

One has  $\deg(K) = 2g - 2 = 0$  for torus reflecting the fact that the canonical form  $\omega = dz$  has no poles or zeros (torus is obtained by identifying the cells of a periodic lattice in complex plane). Correction term vanishes since it would have negative degree for all  $n$  and one has  $l(nP) \in \{1, 1, 2, 3, \dots\}$ .

3.  $g = 2$  case.

For  $n = \deg(D) \geq 2 \times 2 - 1 = 3$  gives  $l(D) = n - 1$  giving for  $n \geq 3$   $l(nP) \in \{2, 3, \dots\}$ . What about  $n = g = 2$ ? For generic points one has  $l(2) = 1$ . There are 6 points at which one has  $l(D) = 2$  so that there is additional meromorphic function having pole of order 2 at this kind of point. These points are fixed points under  $Z_2$  defining hyper-ellipticity. Note that  $g \leq 2$  Riemann surfaces are always hyper-elliptic in the sense that it allows  $Z_2$  as conformal symmetry (see <http://tinyurl.com/y9sdu4o3>).

One has therefore  $l(nP) \in \{1, 1, 1, 2, \dots\}$  for a generic point and  $l(nP) \in \{1, 1, 2, 2, \dots\}$  for 6 points fixed under  $Z_2$ . An interesting question is whether this phenomenon could have physical interpretation in TGD framework.

4.  $g > 2$  case.

For  $g > 2$ .  $l(nP)$  in the range  $\{2, 2g - 2\}$  can depend on point and even on the conformal moduli. There are more special points in which correction term differs from that in the generic case.  $g = 3$  illustrates the situation.  $n \in \{1, 1, 1, 1, 1, 2, \dots\}$  is obtained for a generic point. At special points and for  $n < 3$  there are also other options for  $l(nP)$ . Also the dependence of  $l(nP)$  on moduli emerges for  $g \geq 3$ . The natural guess layman is that these points are fixed points of conformal symmetries. Also now hyper-elliptic surfaces allowing projective  $Z_2$  covering are special. In the general case hyper-ellipticity is not possible.

In TGD framework Weierstrass points(see <http://tinyurl.com/y9wehsm1>) are of special interest physically.

1. My layman guess is that special points known as Weierstrass points (see <http://tinyurl.com/y9wehsm1>) correspond to singularities for projective coverings for which conformal symmetries permute the sheets of the covering. Several points coincide for the covering since a subgroup of conformal symmetries would act trivially on the Weierstrass point.

Note that for  $g > 2$   $Z_2$  covering is not possible except for hyper-elliptic surfaces, and one can wonder whether this relates to the experimental absence of  $g > 2$  fermion families [K1].



Second interesting point is that elementary particles indeed correspond to double sheeted structures from the condition that monopole fluxes flow along closed flux tubes (there are no free magnetic monopoles).

2. There is an obvious analogy with the coverings associated with the cognitive representation defined by the points of space-time surface with coordinates in an extension of rationals [L7, L4] [L6]. Fixed points for a sub-group of Galois group generate singularities at which sheets touch each other. These singular points are physically the most interesting and could carry sparticles. The action of discrete conformal groups restricted to cognitive representation could be represented as the action of Galois group on points of cognitive representation. Cognitive representation would indeed represent!

Remarkably, if the tangent spaces are not parallel for the touching sheets, these points are mapped to several points in  $H$  in  $M^8 - H$  correspondence. If this picture is correct, the hyperelliptic symmetry  $Z_2$  of genera  $g \leq 2$  could give rise to this kind of exceptional singularities for  $g \geq 2$ .

What is worrying that there are two views about twistorial amplitudes. One view relying on the notion of octonionic super-space  $M^8$  [L4] is analogous to that of SUSYs: sparticles can be seen as completely local composites of fermions. Second view relies on imbedding space  $M^4 \times CP_2$  [K19] and on the identification sparticles as non-local many-fermion states at partonic 2-surfaces. These two views could be actually equivalent by  $M^8 - H$  duality.

3. When these singular points are present at partonic 2-surfaces at boundaries of CD and at vertices, the topology of partonic 2-surface is in well-defined sense between  $g$  and  $g+1$  external particles: one has criticality. The polynomials representing external particles indeed satisfy criticality conditions guaranteeing associativity or co-associativity (quantum criticality of TGD Universe is the basic postulate of quantum TGD). At partonic orbits the touching pieces of partonic 2-surface could separate ( $g$ ) or fuse ( $g+1$ ). Could this topological mixing give rise to CKM mixing of fermions [K1, K4, K5]?

#### 3.4.4 RR for algebraic varieties and bundles

RR can be generalized to algebraic varieties (see <http://tinyurl.com/y9asz4qg>). In complex case the real dimension is four so that this generalization is interesting from TGD point of view and will be considered later. The generalization involves rather advanced mathematics such as the notion of sheaf (see <http://tinyurl.com/nudhx06>). Zeros and poles appearing in the divisor are for complex surfaces replaced with 2-D varieties so that the generalization is far from trivial.

The following is brief summary based on Wikipedia article.

1. Genus  $g$  is replaced with algebraic genus and  $deg(D)$  plus correction term is replaced with the intersection number (see <http://tinyurl.com/y7dcffb6>) for  $D$  and  $D - K$ , where  $K$  is the canonical divisor associated with 2-forms, which is also unique apart from linear equivalence. Points of divisor are replaced with 2-varieties.
2. The generalization to complex surfaces (with real dimension equal to 4) reads as

$$\chi(D) = \chi(0) + \frac{1}{2}D \cdot (D - K) . \quad (3.5)$$

$\chi(D)$  is holomorphic Euler characteristic associated with the divisor.  $\chi(0)$  is defined as  $\chi(0) = h_{0,0} - h_{0,1} + h_{0,2}$ , where  $h_{i,j}$  are Betti numbers for holomorphic forms.  $\cdot$  denotes intersection product in cohomology made possibly by Poincare duality.  $K$  is canonical two-form which is a section of determinant bundle having unique divisor (their is linear equivalence due to the possibility to multiply with meromorphic function).

One has  $\chi(0) = 1 + p_a$ , where  $p_a$  is arithmetic genus. Noether's formula gives

$$\chi(0) = \frac{c_1^2 + c_2}{12} = \frac{K \cdot K + e}{12} . \quad (3.6)$$

$c_1^2$  is Chern number and  $e = c_2$  is topological Euler characteristic.

Clearly the information given by  $\chi(D)$  is about Dolbeault homology. For comparison note that RR for curves states  $l(D) - l(K - D) = \chi(D) = \chi(0) + \deg(D)$ .

RR can be also generalized so that it applies to vector bundles. Ordinary RR can be interpreted as applying to a bundle for which the fiber is point. This requires the notion of the inverse bundle defined as a bundle with the property that its direct sum (Whitney sum) with the bundle itself is trivial bundle. One ends up with various characteristic classes, which represent homologically non-trivial forms in the base spaces characterizing the bundle. For instance, the generalizations of RR give information about the dimensions of the spaces of sections of the vector bundle.

Atiyah-Singer index theorem (see <http://tinyurl.com/k6daqco>) deals with so called elliptic operators in compact manifolds and represents a generalization important in recent theoretical physics, in particular gauge theories and string models. The theorem relates analytical index - typically characterizing the dimension for the spectrum of solutions of elliptic operator to a topological index. Elliptic operator is assigned with small perturbations for a given solution of field equations. Perturbations of a given solution of say Yang-Mills equations is a representative example.

## 4 Does $M^8 - H$ duality allow to use the machinery of algebraic geometry?

The machinery of algebraic geometry is extremely powerful. In particular, the number theoretical universality of algebraic geometry implies that same equations make sense for all number fields: this is just what adelic physics [L7] [L6] demands. Therefore it would be extremely nice if one could somehow use this machinery also in TGD framework as it is used in string models. How this could be achieved? There are several guide lines.

1. Twistor lift of TGD [K16, K15, K14, K19] is now a rather well-established idea although a lot of work remains to be done with the details. Twistors were originally introduced in order to be able to use this machinery and involves complexification of Minkowski space  $M^4$  to  $M_c^4$  as an auxiliary tool. Complexification in sufficiently general sense seems to be a necessary auxiliary tool but it cannot be a trick (like Wick rotation) but something fundamental and here complexification at the level of  $M^8$  is suggestive. In the sequel I will use  $M^4$  for  $M_c^4$  and  $M^8$  for  $M_c^8$  unless it is necessary to emphasize that  $M_c^8$  is in question. The essential point is that the Euclidian metric is complexified and it reduces to a real metric in various sub-spaces defining besides Euclidian space also Minkowski spaces with varying signature when the complex coordinates are real or imaginary.
2. If  $M^8 - H$  duality holds true, one can solve field equations in  $M^8 = M^4 \times E^8$  by assuming that either the tangent space or normal space of the space-time surface  $X^4$  is associative (quaternionic) at each point and contains preferred  $M^2$  in its tangent space.  $M^2$  could depend on  $x$  but  $M^2(x)$ 's should integrate to a 2-surface. This allows to map space-time surface  $M^8$  to a surface in  $M^4 \times CP_2$  since tangent spaces are parameterized by points of  $CP_2$  and  $CP_2$  takes the role of moduli space. The image of tangent space as point of  $CP_2$  is same irrespective of whether one has quaternions or complexified quaternions ( $H_c$ ).

It came a surprise that associativity/co-associativity is possible only if the space-time surface is critical in the sense that some gradients of 8 complex components of the octonionic polynomial  $P$  vanish without posing them as additional conditions reducing thus the dimension of the space-time surface. This occurs when the coefficients of  $P$  satisfy additional conditions. One obtains associative/co-associative space-time regions and regions without either property and they correspond nicely to two solution types for the twistor lift of Kähler action.

3. Contrary to the original expectations,  $M^4 \subset M_c^8$  must be identified as co-associative (co-quaternionic) subspace so that  $E^4$  is the associative/quaternionic sub-space. This allows to have light-cone boundary as the counterpart of point-like singularity in ordinary algebraic geometry and also allows to understand the emergence of CDs and ZEO.

**Remark:** A useful convention to be used in the sequel.  $RE(o)$  and  $IM(o)$  denote the real and imaginary parts of the octonion in the decomposition  $o = RE(o) + IM(o)I_4$  and  $Re(o)$  and  $Im(o)$  its real number valued and purely imaginary parts in the usual decomposition.

The problems related to the signature of  $M^4$  have been a longstanding head-ache of  $M^8$  duality.

1. The intuitive vision has been that the problems can be solved by replacing  $M^8$  with its complexification  $M_c^8$  identifiable as complexified octonions  $o$ . This requires introduction of imaginary unit  $i$  commuting with the octonionic units  $E^k \leftrightarrow (1, I_1, \dots, I_7)$ . The real octonionic components are thus replaced with ordinary complex numbers  $z_i = x_i + iy_i$ .
2. Importantly, complex conjugation  $o \rightarrow \bar{o}$  changes only the sign of  $I_i$  but *not!* that of  $i$  so that the octonionic inner product  $(o_1, o_2) = o_1 \bar{o}_2 = o_1^k o_2^l \delta_{k,l}$  becomes complex valued. Norm is equal to  $O\bar{O} = \sum_i z_i^2$ . Both norm and inner product are in general complex valued and real valued only in sub-spaces in which octonionic coordinates are real or imaginary. Sub-spaces have all possible signatures of metric. These sub-spaces are not closed under multiplication and this has been an obstacle in the earlier attempts based on the notion of octonion analyticity. This argument applies also to quaternions and one obtains signatures  $(1, 1, 1, 1)$ ,  $(1, 1, 1, -1)$ ,  $(1, 1, -1, -1)$ , and  $(1, -1, -1, -1)$ . Why just the usual Minkowskian signature  $(1, -1, -1, -1)$  is physical, should be understood.

The convention consistent with that used in TGD corresponds to a negative length squared for space-like vectors and positive for time-like vectors. This gives  $m = (o^0, io^1, \dots, io^7)$  with real  $o^k$ . The projection  $M_c^8 \rightarrow M^8$  defines the projection of  $X_c^4 \subset M_c^8$  to  $X^4 \subset M^8$  serving as the pre-image of  $X^4 \subset M^8$  in  $M^8 - H$  correspondence.

3.  $o$  is not field anymore as is clear from the fact that  $1/o = \bar{o}/o\bar{o}$  is formally infinite in Minkowskian sub-spaces, when octonion defines a light-like vector.  $o$  (and  $H_c$ ) remains however a ring so that sum and products are well-defined but division can lead to problems unless one stays inside 7+7-dimensional light-cone with  $Re(o\bar{o}) > 0$  ( $Re(q\bar{q}) > 0$ ).

Although the number field structure is lost, one can still define polynomials needed to define algebraic varieties by requiring their simultaneous vanishing and rational functions make sense inside the light-cone. Also rational functions can be defined but poles are replaced with light-cones in Minkowskian section. Algebraic geometry would thus be forced by the complexification of octonions. This looks to me highly non-trivial! The extension of zeros and poles to light-cones making propagation possible could be a good reason for why Minkowskian signature is physical. Interestingly, the allowed octonionic momenta are light-like quaternions [K19].

4. An interesting question is whether ZEO and the emergence of CDs relates to the failure of field property. It seems now clear that CDs must be assigned even with elementary particles. I have asked whether they could form an analog for the covering of manifold by coordinate patches (in TGD inspired theory of consciousness CDs would be correlates for perceptive fields for conscious entities assignable to CDs [L8]). These observations encourage to ask whether the tips of CD should correspond to a pair formed by two poles/two zeros or by pole and zero assignable to positive and negative energy states.

It turns out that the space-time surfaces in the interior of CD would naturally correspond to non-associative surfaces and only their 3-D boundaries would have associative 4-D tangent spaces allowing mapping to  $H$  by  $M^8$ -duality, which is enough by holography.

5. The relationship between light-like 3-surface bounding Minkowskian and Euclidian space-time regions and light-like boundaries of CDs is interesting. Could also the partonic orbits be understood a singularities of octonionic polynomials with  $IM(P) = RE(P) = 0$ ?

#### 4.1 What does one really mean with $M^8 - H$ duality?

The original proposal was that  $M^8$  duality should map the associative tangent/normal planes of associative/co-associative space-time surface containing preferred  $M^2$ , call it  $M_0^2$ , to  $CP_2$ : the map read as  $(m, e) \in M^4 \times E^4 \rightarrow (m, s) \in M^4 \times CP_2$ . Eventually it became clear that the choice of  $M^2$  can depend on position with  $M^2(x)$  forming an integrable distribution to  $CP_2$ : this would define

what I have called Hamilton-Jacobi structures [K12]. String like objects have minimal surface as  $M^4$  projection for almost any general coordinate invariant action, and internal consistency requires that  $M^2(x)$  integrate to a minimal surface. The details are however not understood well enough.

1.  $M^4$  coordinate would correspond simply to projection to a fixed  $M_0^4$  in the decomposition  $M^8 = M_0^4 \times E_0^4$ . One can however challenge this interpretation. How  $M_0^4$  is chosen? Is it possible to choose it uniquely? Could also  $M^4$  coordinates represent moduli analogous to  $CP_2$  coordinates? What about ZEO?

There is an elegant general manner to formulate the choice of  $M_0^4$  at the level of  $M^8$ . The complexified quaternionic sub-spaces of  $M_c^8$  ( $M^8$ ) are parameterized by moduli space defining the quaternionic moduli. The moduli space in question is  $CP_2$ . The choice of  $M_0^4$  corresponds to fixing of the quaternionic moduli by fixing a point of  $CP_2$ .

**Warning:** Note that one should be very careful in distinguishing between quaternionic as sub-spaces of  $M^8$  and as the tangent space  $M^8$  of given point of  $M^8$ .

2. One can ask whether there could be a connection with ZEO, where CDs play a key role. Indeed, the complexified Minkowski inner product means that the complexified octonions (quaternions) inside  $M_c^8$  ( $M_c^4$ ) have inverse only inside 7-D (4-D) complexified light-cone and this would motivate the restriction of space-time surfaces inside future or past light-cone or both but not yet force CD.

If one allows rational functions and even meromorphic functions of octonionic or quaternionic variable, one could consider the possibility of restricting the space-time surface defined as their zeros to a maximally sized region containing no poles.

3. Consider complexified quaternions  $H_c$ . Poles  $(q\bar{q})^{-n}$ ,  $n \geq 1$  would correspond  $M^4$  light-cone boundaries since  $q\bar{q} = 0$  at them. Also zeros  $q\bar{q} = 0$ , for  $n \geq 1$  correspond to light-like boundaries. Could one have two poles with with time-like distance defining CD or a pair of pole and zero?

There is also a possible connection with the notion of infinite primes [K6]. The notion of infinite prime leads to the proposal that rationals defined as ratios of infinite integers but having unit real norm (and also p-adic norms) could correspond pairs of positive and negative energy states with identical total quantum numbers and located at opposite boundaries of CD. Infinite rationals can be mapped to rational functions. Could positive energy states correspond to the numerators with zeros at second boundary of CD and negative energy states to denominators with zeros at opposite boundary of CD?

#### 4.1.1 Is the choice of the pair $(M_0^2, M_0^4)$ consistent with the properties of known extremals in $H$

It should be made clear that the notion of associativity/co-associativity (quaternionicity/co-quaternionicity) of the tangent/normal space need not make sense at the level of  $H$ . I shall however study this working hypothesis in the sequel.

The choice of the pair  $(M_0^2, M_0^4)$  means choosing preferred co-commutative (commutative) sub-space  $M_0^2$  of  $M^8$  defining a subspace of fixed co-quaternionic (quaternionic) sub-space  $M_0^4 \subset M^8$ .

**Remark:**  $M^4$  should indeed be the co-associative/co-quaternionic subspace of  $M^8$  if the argument about emergence of CDs is accepted and if  $M^8 - H$  correspondence maps associative to associative and co-associative to co-associative.

$M_0^4$  in turn contains preferred  $M_0^2$  defining co-commutative (hyper-complex) structure. Both  $M_0^2$  and  $M_0^4$  are needed in order to label the choice by  $CP_2$  point (that is as a point of Grassmanian).

Is the projection to a fixed factor  $M_0^4 \subset M_0^4 \times E^4$  as a choice of co-quaternionic moduli consistent with what we know about the extremals of twistor lift of Kähler action in  $H$ ? How could one fix  $M_0^4$  from the data about the extremal in  $H$ ? One can make similar equations about the choice of  $M_0^2$  as a fixed co-complex moduli characterized by a unit quaternion. Note that this choice is expected to relate closely to the twistor structure and Kähler structure.

It is best to check the proposal for the known extremals in  $H$  [K12]. Consider first  $CP_2$  type extremals for which  $M^4$  projection is a piece of light-like geodesic.

1. The  $CP_2$  projection for the image of  $X^4 \subset M^8$  differs from single point only if the tangent space isomorphic to  $M^4$  and parameterized by  $CP_2$  point varies. Consider  $CP_2$  type extremals for the twistor lift of Kähler action [?] in  $H$  having light-like geodesic as  $M^4$  projection as an example. The light-like geodesic defines a light-like vector in the tangent space of  $CP_2$  type extremal. This light-like vector together with its dual spans fixed  $M^2$ , which however does not belong to the tangent space so that associative surface would not be in question.

What about co-associativity or associativity (the latter is favored by above argument)? This property should hold true for the pre-image of  $CP_2$  type extremal in  $M^8$  but I am not able to say anything about this. It is questionable to require this property at the level  $H$  but one can of course look what it would give.

What about associativity for  $CP_2$  tangent space? The normal space of  $CP_2$  type extremal is 3-D (!) since the only the light-like tangent vector of the geodesic and 2 vectors orthogonal to it are orthogonal to  $CP_2$  tangent vectors. For Euclidian signature this would mean that tangent space is 5-D and cannot be associative but now the tangent space is 4-D. Can one still say that tangent space is associative. The co-associativity of the tangent space makes sense trivially. Can one conclude that  $CP_2$  is co-associative.

The associativity for  $CP_2$  tangent space might make sense since the tangent space is 4-D. The light-like vector  $k$  defines  $M_0^2$ . The associativity conditions involving two tangent space vectors of  $CP_2$  and the light-like vector  $k$  contracted with the corresponding octonion components. The contributions from the components of  $k$  to the associator should cancel each other. Since one can change the relative sign of the components of  $k$ , this mechanism does not seem to work for all components. Hence associativity cannot hold true. Neither does  $M_0^2$  belong to the normal space since  $k$  and its dual are not orthogonal.

Could one conclude that  $CP_2$  type extremal is co-associative in accordance with the original belief thanks to the light-like projection to  $M^4$ ? This does not conform with what the singularity considerations for the octonionic polynomials would suggest. Or is it simply not correct to try to apply associativity at the level of  $H$ . Or does  $M^8 - H$  correspondence map associative tangent spaces to co-associative ones?

2. The normal space  $M^4$  of  $CP_2$  type extremal have all orientations characterized by its  $CP_2$  projection. The normal space must contain the  $M_0^2$  determined by the tangent of the light-like geodesic and this is indeed the case. Note that  $CP_2$  type extremals cannot have entire  $CP_2$  as  $CP_2$  projection: they necessarily have hole at either end, which would be naturally be at the boundary of CD.

$CP_2$  type extremals seem to be consistent with  $M^8 - H$  correspondence. It however seems that one cannot fix the choice of  $M_0^4$  uniquely in terms of the properties of the extremal. There is a moduli space for  $M_0^4$ :s defined by  $CP_2$  and obviously codes for moduli for quaternion structures in octonionic space. The distributions of  $M^2(x)$  (minimal surfaces) would code for quaternion structures (decomposition of octonionic coordinates to quaternionic coordinates in turn decomposing to pairs of complex coordinates).

Consider next the associativity condition for cosmic strings in  $X^2 \times Y^2 \subset M^4 \times CP_2$ . Now  $CP_2$  projection is 2-D complex surfaces and  $M^4$  projection is minimal surface. Situation is clearly associative. How unique the choice of  $M_0^4$  is now?

1. Now  $M^2(x)$  depends on position but  $M^2(x)$ :s define an integrable distribution defining string orbit  $X^2$  as a minimal surface.  $M_0^4$  must contain all surfaces  $M^2(x)$ , which would fix  $M_0^4$  to a high degree for complex enough cosmic strings.
2. Each point of  $X^2$  corresponds to the same partonic surface  $Y^2 \subset CP_2$  labelling the tangent spaces for its pre-image in  $M^8$ . All the tangent surfaces  $M^2(x) \times E^2(y)$  for  $X^2 \times Y^2 \subset M^8$  share only  $M^2(x) \subset M_0^4$ .  $M_0^4$  must contain all tangent spaces  $M^2(x)$  and the inverse image of  $Y^2 \subset CP_2$  must belong to the orthogonal complement  $E^4$  of  $M_0^4$ . This is completely analogous with the condition  $X^2 = X^2 \times Y^2 \subset M^4 \times CP_2$ .

Consider a decomposition  $M^8 = M_0^4 \times E^4$ ,  $M_0^4 = M_0^2 \times E_0^2$ . If the inverse image of  $Y^2$  at point  $x$  belongs to  $E^4$ , the  $M_0^4$  projection belongs to  $M_0^4$  also in  $M^8$ . If this does not pose

any condition on the tangent spaces assignable to the points of  $Y^2$  defining points of  $CP_2$ , there are no problems. What could happen that the tangent spaces assignable to  $Y^2$  could force the projection of the inverse image of  $Y^2$  to intersect  $M_0^4$ .

One should also understand massless extremals (MEs). How to choose  $M_0^4$  in this case?

1. MEs are given as zeros of arbitrary functions of  $CP_2$  coordinates and 2  $M^4$  coordinates  $u$  and  $v$  representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant and define  $M_0^4 = M_0^2 \times E_0^2$  decomposition everywhere so that  $M_0^4$  is uniquely defined. Same applies also when the directions are not constant. In the general case light-like direction would define the local tangent plane of string world sheet and local polarization plane. Since the dimension of  $M^4$  projection is 4 there seems to be no problems involved.
2. Tangent plane of  $X^4$  is parameterized by  $CP_2$  coordinates depending on 2 coordinates  $u$  and  $v$ . The surface  $X^4 \subset M^8$  must be graph for a map  $M_0^4 \rightarrow E^4$  so that a 2-parameter deformation of  $M_0^4$  as tangent plane is in question. The distribution of tangent planes of  $X^4 \subset M^8$  is 2-D as is also the  $CP_2$  projection in  $H$ .

To sum up, the original vision about the associativity properties of the known extremals at level of  $H$  survives. On the other hand, CDs emerge if  $M^4$  corresponds to the co-associative part of  $O$ . Does this mean that  $M^8 - H$  correspondence maps associative to co-associative by multiplying the quaternionic tangent space in  $M^8$  by  $I_4$  to get that in  $H$  and vice versa or that the notions of associative and co-associative do not make sense at the level of  $H$ ? This does not affect the correspondence since the same  $CP_2$  point parametrizes both associative tangent space and its complement.

#### 4.1.2 Space-time surfaces as co-dimension 4 algebraic varieties defined by the vanishing of real or imaginary part of octonionic polynomial?

If the theory intended to be a theory of everything, the solution ansatz for the field equations defining space-time surfaces should be ambitious enough: nothing less than a general solution of field equations should be in question.

1. One cannot exclude the possibility that all analytic functions of complexified octonionic variable with real Taylor or even Laurent coefficients. These would would a commutative and associative algebra. Space-time surfaces would be identified as their zero loci. This option is however number theoretically attractive and can also leads to problems with adelic physics. Since Taylor series at rational point need not anymore give a rational value.
2. Polynomials of complexified octonion variable  $o$  with real coefficients define the simplest option but also rational functions formed as ratios of this kind of polynomials must be considered. Polynomials form a non-associative ring allowing sum, product, and functional decomposition as basic operations. If the coefficients  $o_n$  of polynomials are complex numbers  $o_n = a_n + ib_n$ ,  $a_n, b_n$  real, where  $i$  refers to the commutative imaginary unit complexifying the octonions, the ring is associative. It is essential to allow only powers  $o^n$  (or  $(o - o_0)^n$  with  $o_0 = a_0 + ib_0$ ,  $a_0, b_0$  real numbers). Physically this means that a preferred time axis is fixed. This time axis could connect the tips of CD in ZEO.

One can write

$$P(o) = \sum_k p_k o^k \equiv RE(P)(q_1, q_2, \bar{q}_1, \bar{q}_2) + IM(P)(q_1, q_2, \bar{q}_1, \bar{q}_2) \times I_4, \quad p_k \text{ real}, \quad (4.1)$$

where the notations

$$o = q_1 + q_2 I_4, \quad q_i = z_i^1 + z_i^2 I_2, \quad \bar{q}_i = z_i^1 - z_i^2 I_2, \quad z_i^j = x_i^j + iy_i^j \quad (4.2)$$

Note that the conjugation does *not* change the sign of  $i$ . Due to the non-commutativity of octonions  $P^i$  as functions of quaternions are in general *not* analytic in the sense that they would be polynomials of  $q_i$  with real coefficients! They are however analytic functions of  $z_i$ . The real and imaginary parts of  $x_i^j$  correspond to Minkowskian and Euclidian signatures.

In adelic physics coefficients  $o_n$  of the octonionic polynomials define WCW coordinates and should be rational numbers or rationals in the extension of rationals defining the adèle. The polynomials form an associative algebra since associativity holds for powers  $o^n$  multiplied by real number. Thus complex analyticity crucial in algebraic geometry would be a key element of adelic physics.

3. If the preferred extremals correspond to the associative algebra formed by these polynomials, one could construct a completely general solution of the field equations as zero loci of their real or imaginary parts and build up of new solutions using algebra operation sum, product, and functional decomposition. One could identify space-time regions as associative or co-associative algebraic varieties in terms of these polynomials and they would form an algebra.

The motivation for this dream comes from 2-D electrostatics, where conducting surfaces correspond to curves at which the real part  $u$  or imaginary part  $v$  of analytic function  $w = f(z) = u + iv$  vanishes. In electrostatics curves form families with curves orthogonal to each other locally and the map  $w = u + iv \rightarrow v - iu$  defines a duality in which curves of constant potential and the curves defining their normal vectors are mapped to each other.

1. The generalization to the recent situation would be vanishing of the imaginary part  $IM(P)$  or real part  $RE(P)$  of the octonionic polynomial, where real and imaginary parts are defined via  $o = q_c^1 + q_c^2 I_4$ . One can consider also the possibility that imaginary or real part has constant value  $c$  are restricted to be rational so that one can regard the constant value set also as zero set for a polynomial with constant shift. Note that the rationals could be also complexified by addition of  $i$ . One would have

$$RE(P)(z_i^k) \quad \text{or} \quad IM(P)(z_i^k) = c, \quad c = c_0 \text{ rational} . \quad (4.3)$$

$c_0$  must be real. These two options should correspond to the situations in which tangent space or normal space is associative (associativity/co-associativity). Complexified space-time surfaces  $X_c^4$  corresponding to different constant values  $c$  of imaginary or real part (with respect to  $i$ ) would define foliations of  $M_c^8$  by locally orthogonal 4-dimensional surfaces in  $M_c^8$  such that normal space for surface  $X_c^4$  would be tangent space for its co-surface.

CDs and ZEO emerges naturally if the  $IM(o)$  corresponds to co-quaternionic part of octonion.

2. It must be noticed that one has moduli space for the quaternionic structures even when  $M_0^4$  is fixed. The simplest choices of complexified quaternionic space  $H_c = M_{c,0}^4$  containing preferred complex plane  $M_{c,0}^2$  and its orthogonal complement are parameterized by  $CP_2$ . More complex choices are characterized by the choice of distribution of  $M^2(x)$  integrable to (presumably minimal) 2-surface in  $M^4$ . Also the choice of the origin matters as found and one has preferred coordinates. Also the 8-D Lorentz boosts give rise to further quaternionic moduli. The physically interesting question concerns the interpretation of space-time surfaces with different moduli. For instance, under which conditions they can interact?

The proposal has several extremely nice features.

1. Single real valued polynomial of real coordinate extended to octonionic polynomial and fixed by real coefficients in extension of rationals would determine space-time surfaces.
2. The notion of analyticity needed in concrete equations is just the ordinary complex analyticity forced by the octonionic complexification: there is no need for the application to have left- or right quaternion analyticity since quaternionic derivatives are not needed. Algebraically

one has the most obvious guess for the counterpart of real analyticity for polynomials generalized to octonionic framework and there is no need for the quaternionic generalization of Cauchy-Riemann equations [A10, A4] [A10, A4] (<http://tinyurl.com/yb8134b5>) plagued by the problems with the definition of differentiation in non-commutative and non-associative context. There would be no problems with non-associativity and non-commutativity thanks to commutativity of complex coordinates with octonionic units.

3. The vanishing of the real or imaginary part gives rise to 4 conditions for 8 complex coordinates  $z_1^k$  and  $z_2^k$  allowing to solve  $z_2^k$  as algebraic functions  $z_2^k = f^k(z_1^l)$  or vice versa. The conditions would reduce to algebraic geometry in complex co-dimension  $d_c = 4$  and all methods and concepts of algebraic geometry can be used! Algebraic geometry would become part of TGD as it is part of M-theory too.

## 4.2 Is the associativity of tangent-/normal spaces really achieved?

The non-trivial challenge is to prove that the tangent/normal spaces are indeed associative for the two options. The surfaces  $X_c^4$  are indeed associative/co-associative if one considers the *internal* geometry since points are in  $M_c^4$  or its orthogonal complement.

One should however prove that  $X_c^4$  are also associative *as sub-manifolds* of  $O$  and therefore have quaternionic tangent space or normal space at each point parameterized by a point of  $CP_2$  in the case that tangent space containing position dependent  $M_c^2$ , which integrate to what might be called a 2-D complexified string world sheet inside  $M_c^4$ .

1. The first thing to notice that associativity and quaternionicity need not be identical concepts. Any surface with complex dimension  $d < 4$  in  $O$  is associative and any surface with dimension  $d > 4$  co-associative. Quaternionic and co-quaternionic surfaces are 4-D by definition. One can of course ask whether one should consider a generalization of brane hierarchy of M-theory also in TGD context and allow associativity in its most general sense. In fact, the study of singularity of  $o^2$  shows that 6 and 5-dimensional surfaces are allowed for which the only interpretation would be as co-associative spaces. This exceptional situation is due to the additional symmetries increasing the dimension of the zero locus.
2. One has clearly quaternionicity at the level of  $o$  obtained by putting  $Y = 0$  and at the level of the tangent space for the resulting surface. The tangent space should be quaternionic. The Jacobian of the map defined by  $P$  is such that it takes fixed quaternionic subspace  $H_c \rightarrow M_{0,c}^4$  of  $O$  to a quaternionic tangent space of  $X^4$ . The Jacobian applied to the vectors of  $H_c$  gives the octonionic tangent vectors and they should span a quaternionic sub-space.
3. The notion of quaternionic surface is rigorous.  $M^8 - H$  correspondence could be actually interpreted in terms of the construction of quaternionic surface in  $M^8$ . One has 4-D integrable distribution of quaternionic planes in  $O$  with given quaternion structure labelled by points of  $CP_2$  and has representation at the level of  $H$  as space-time surface and should be preferred extremals. These quaternion planes should integrate to a slicing by 4-surfaces and their duals. One obtains this slicing by fixing the values 4 of the suitably defined octonionic coordinates  $P^i$ ,  $i = 1, \dots, 8$ , to a real constants depending on the surface of the slicing. This gives a space-time surfaces for which tangent space-spaces or normal spaces are quaternionic.

The first guess for these coordinates  $P^i$  be as real or imaginary parts of real polynomials  $P(o)$ . But how to prove and understand this?

Could the following argument be more than wishful thinking?

1. In complex case an analytic function  $w(z) = u + iv$  of  $z = x + iy$  mediates a map between complex planes  $Z$  and  $W$ . One can interpret the imaginary unit appearing in  $w$  locally as a tangent vector along  $u = \text{constant}$  coordinate line.
2. One can interpret also octonionic polynomials with real coefficients as mediating a map from octonionic plane  $O$  to second octonionic plane, call it  $W$ . The decomposition  $P = P^{(1)} + P^{(2)}$  would have interpretation in terms of coordinates of  $W$  with coordinate lines representing quaternions and co-quaternions.



3. This would suggest that the quaternionic coordinate lines in  $W$  can be identified as coordinate curves in  $O$  - that space-time surfaces - which are quaternionic/co-quaternionic surfaces for  $P^1 = \text{constant}/P^2 = \text{constant}$  lines. One would have a representation of the same thing in two spaces, and if sameness includes also quaternionicity/co-quaternionicity as attributes, then also associativity and co-associativity should hold true.

The most reasonable approach is based on generality. Associativity/quaternionicity means a slicing of octonion space by orthogonal quaternionic and co-quaternionic 4-D surfaces defined by constant value surfaces of octonionic polynomial with real coefficients. This slicing should make sense also for quaternions: one should have a slicing by complex and co-complex (commutative/co-commutative) surfaces and in TGD string world sheets and partonic 2-surfaces assignable to Hamilton-Jacobi structure would define this kind of slicing. In the case of complex numbers one has a slicing in terms of constant value curves for real and imaginary parts of analytic function and Cauchy-Riemann equations should define the property and co-property. The first guess that the tangent space of the curve is real or imaginary is wrong.

#### 4.2.1 Could associativity and commutativity conditions be seen as a generalization of Cauchy-Riemann conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The “Whatever it is” cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions  $D = 2^k$ ,  $k = 1, 2, 3$ :  $k$ -linearity with  $k = 1, 2, 3$ !

One can continue the hierarchy of division algebras by assuming only algebra property by using Cayley-Dickson construction (see <http://tinyurl.com/ybuy1a2k>) by adding repeatedly a non-commuting imaginary unit to the structure already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has  $x^m x^n = x^{m+n}$ . For instance, sedenions define 16-D algebra. Could this hierarchy correspond to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

#### 4.2.2 Complex curves in real plane cannot have real tangent space

Going from octonions to quaternions to complex numbers, could constant value curves of real and imaginary parts of ordinary analytic function in complex plane make sense? The curves  $u = 0$  and  $v = 0$  of functions  $f(z) = u + iv$ ,  $z = x + iy$  define a slicing of plane by orthogonal curves completely analogous to its octonionic and quaternionic variants. Can one say that the tangent vectors for these curves are real/imaginary? For  $u = 0$  these curves have tangent  $\partial_x u + i\partial_y u$ , which is not real unless one has  $f(z) = k(x + iy)$ ,  $k$  real.

Reality condition is clearly too strong. In fact, it is the well-ordering of the points of the 1-dimensional curve, which is the property in question and lost for complex numbers and regained at  $u = 0$  and  $v = 0$  curves. The reasonable interpretation is in terms of hierarchy of conditions multilinear in the gradients of coordinates proposed above and linear Cauchy-Riemann conditions is the only option in the case of complex plane. What is special in these curves that the tangent vectors define flows which by Cauchy-Riemann conditions are divergenceless and irrotational locally.

Pessimistic would conclude that since the conjecture fails except for linear polynomials in complex case, it fails also in the case of quaternions and octonions. For quaternionic polynomial  $q^2$  the conditions are however satisfied and it turns out that the resulting conditions make sense also in the general case. Optimistic would argue that reality condition is not analogous to commutativity and associativity so that this example tells nothing. Less enthusiastic optimist might admit that the reality condition is a natural generalization to complex case but that the conjecture might be true

only for a restricted set of polynomials - in complex case of for  $f(z) = kz$ ,  $k$  real. In quaternionic and octonionic case but hopefully for a larger set of polynomials with real coefficients, maybe even all polynomials with real coefficients.

### 4.2.3 Associativity and commutativity conditions as a generalization of Cauchy-Riemann conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The “whatever-it-is” cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions  $D = 2^k$ ,  $k = 1, 2, 3$ :  $k$ -linearity with  $k = 1, 2, 3$ !

One can continue the hierarchy of number fields by assuming only algebra property by adding additional imaginary units as done in Cayley-Hamilton construction (see <http://tinyurl.com/ybuy1a2k>) by adding repeatedly a non-commuting imaginary unit to the algebra already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has  $x^m x^n = x^{m+n}$ . For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions? Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

One would have also a nice physical interpretation: in the case of quaternions one would have “quaternionic conformal invariance” as conformal invariances inside string world sheets and partonic 2-surfaces in a nice agreement with basic vision about TGD. At the level of octonions would have “quaternionic conformal invariance” inside space-time surfaces and their duals. What selects the preferred commutative or co-commutative surfaces is of course an interesting problem. Is a gauge choice in question? Are these surfaces selected by some special property such as singular character? Or does one have wave function in the set of these surfaces for a given space-time surface?

### 4.2.4 Could quaternionic polynomials define complex and co-complex surfaces in $H_c$ ?

What about complex and co-complex (commutative/co-commutative) surfaces in the space of quaternions? One would have a slicing of the quaternionic space by pairs of complex and co-complex surfaces and would have natural identification as quaternion/Hamilton-Jacobi structure and relate to the decomposition of space-time to string world sheets and partonic 2-surfaces. Now the condition of associativity would be replaced with commutativity.

1. In the quaternionic case the tangent vectors of the 2-D complex sub-variety would be commuting. Can this be the case for the zero loci real polynomials  $P(q)$  with  $IM(P) = 0$  or  $RE(P) = 0$ ? In this case the commutativity condition is that the tangent vectors have imaginary parts (as quaternions) proportional to each other but can have different real parts. The vanishing of cross product is the condition now and involves only two vectors whereas associativity condition involves 3 vectors and is more difficult.
2. The tangent vectors of a commutative 2-surface commute:  $[t^1, t^2] = 0$ . The commutator reduces to the vanishing of the cross product for the imaginary parts:

$$Im(t^1) \times Im(t^2) = 0 \quad . \quad (4.4)$$

3. Expressing  $z_1^i$  as a function of  $z_2^k$  and using  $(z_1^i, z_2^k)$  as coordinates in quaternionic space, the tangent vectors in quaternionic spaces can be written in terms of partial derivatives  $\partial z_1^i / \partial z_2^k$  as

$$t_k^i = \left( \frac{\partial z_1^i}{\partial z_2^k}, \delta_k^i \right) , \quad (4.5)$$

Here the first part corresponds to  $RE(t^i)$  as quaternion and second part to  $IM(t^i)$  as quaternion.

The condition that the vectors are parallel implies

$$\frac{\partial z_1^i}{\partial z_2^k} = 0 . \quad (4.6)$$

At the commutative 2-surface  $X^2$   $z_1^1$  is constant and  $z_1^2$  is a function of  $z_2^1$  and  $z_2^2$ . One would have a graph of a function  $z_1^2 = f_2(z_2^k)$  at  $X^2$  but not elsewhere. One could regard  $z_1^1$  as an extremum of a function  $z_1^1 = f_1(z_2^k)$ .

How to interpret this result?

1. In the generic case this condition eliminates 1 dimension so that that 2-D surface would reduce to a 1-D curve.
2. If one poses constraints on the coefficients of  $P(q)$  analogous to the conditions forcing the potential function for say cusp catastrophe to have degenerate extrema at the boundaries of the catastrophe one can get 2-D solution. For these values of parameters the conditions would be equivalent with  $RE(P) = 0$  or  $IM(P) = 0$  conditions.

The vanishing of the gradient of  $z_1^1$  would indeed correspond in the case of cups catastrophe to the condition for the co-incidence of two roots of the behavior variable  $x$  as extremum of potential function  $V(x, a, b)$  fixing the control variable  $a$  as function of  $b$ .

This would pose constraints on the coefficients of  $P$  not all polynomials would be allowed. This kind of conditions would realize the idea of quantum criticality of TGD at the level of quaternion polynomials. This option is attractive if realizable also at the level of octonion polynomials. This turns out to be the case.

3. One would thus have two kinds of commutative/co-commutative surfaces. The generic 1-D surfaces and 2-D ones which are commutative/commutative and critical and assignable to string world sheets and partonic 2-surfaces. 1-D surfaces would correspond to fermion lines at the orbits of partonic 2-surfaces appearing in the twistor amplitudes in the interaction regions defined by CDS. 2-D surfaces would correspond to the orbits of fermionic strings connecting point-like fermions at their ends and serving as correlates for bound state entanglement for external fermions arriving into CD. This picture would allow also to understand why just some string world sheets and partonic 2-surfaces are selected.

The simplest manner to kill the proposal is to look for  $P = q^2$  and  $RE(P(q^2)) = 0$  surface. In this case this condition is indeed satisfied. One has

$$\begin{aligned} RE(P) &= X^1 + X^2 I_1 , \\ X^1 &= (z_1^1)^2 - (z_1^2)^2 + (z_2^1)^2 - (z_2^2)^2 , \quad X^2 = 2z_1^1 z_1^2 I_1 , \\ IM(P) &= Y^1 + Y^2 I_1 , \\ Y^1 &= (z_2^1 + \overline{z_2^1}) z_1^1 , \quad Y^2 = (z_2^2 + \overline{z_2^2}) z_1^2 \end{aligned} \quad (4.7)$$

$X^2 = 0$  gives  $z_1^1 z_1^2 = 0$  so that one has either  $z_1^1 = 0$  or  $z_1^2 = 0$ .  $X^1 = 0$  gives for  $z_1^1 = 0$   $z_1^2 = \pm \sqrt{(z_2^1)^2 + (z_2^2)^2}$ .

The partial derivative  $\partial z_1^1 / \partial z_2^k$  is from implicit function theorem - following from the vanishing of the differential  $d(RE(P))$  along the surface - proportional  $\partial X^1 / \partial z_2^k$ , but vanishes as required.

Clearly, the quaternionic variant of the proposal survives in the simplest case its simplest test. 2-D character of the surface would be due to the criticality of  $q^2$  making it possible to satisfy the conditions without the reduction of dimension.

#### 4.2.5 Explicit form of associativity/quaternionicity conditions

Consider now the explicit conditions for associativity in the octonionic case.

1. One should calculate the octonionic tangent (normal) vectors  $t^i$  for  $X = 0$  in associative ( $Y = 0$  in co-associative case) and show that there associators  $Ass(t^i, t^j, t^k)$  vanish for all possible or all possible combinations  $i, j, k$ . In other words, one that one has

$$Ass(t^i, t^j, t^k) = 0 \quad , \quad i, j, k \in \{1, \dots, 4\} \quad , \quad Ass(a, b, c) \equiv (ab)c - a(bc) \quad . \quad (4.8)$$

One can cast the condition to simpler form by expressing  $t^i$  as octonionic vectors  $t_k^i E^k$ :

$$\begin{aligned} Ass(E^a, E^b, E^c) &\equiv f^{abcd} E_d \quad , \quad a, b, c, d \in \{1, \dots, 7\} \quad , \\ f^{abcd} &= \epsilon^{abe} \epsilon_e^{cd} - \epsilon^{aed} \epsilon_e^{bc} = 2\epsilon^{abe} \epsilon_e^{cd} \quad . \end{aligned} \quad (4.9)$$

The permutation symbols for a given triplet  $i, j, k$  are structure constants for quaternionic inner product and completely antisymmetric (see <http://tinyurl.com/p42tqsq>)..  $\epsilon_{ijk} = 1$  is true for the seven triplets 123, 145, 176, 246, 257, 347, 365 defining quaternionic sub-spaces with 1-D intersections. The anti-associativity condition  $(E_i E_j) E_k = -(E_i E_j) E_k$  holds true so that one has obtains the simpler expression for  $f^{ijk}$  having values  $\pm 2$ .

Using this representation  $Ass(t^i, t^j, t^k)$  reduces to 7 conditions for each triplet:

$$t_r^i t_s^j t_t^k f^{rstu} = 0 \quad , \quad i, j, k \in \{1, \dots, 4\} \quad , \quad r, s, t, u \in \{1, \dots, 7\} \quad . \quad (4.10)$$

2. If the vanishing condition  $X = 0$  or  $Y = 0$  is crucial for associativity then every polynomial is its own case to be studied separately and a general principle behind associativity should be identified: the proposal is as a non-linear generalization of Cauchy-Riemann conditions. As the following little calculation shows, the vanishing condition indeed appears as one calculates partial derivatives  $\partial z_1^k / \partial z_2^l$  in the expression for the tangent vectors of the surface deduced from the vanishing gradient of  $X$  or  $Y$ .
3. I have proposed the octonionic polynomial ansatz already earlier but failed to prove that it gives associative tangent or normal spaces. Besides the intuitive geometric argument I failed to notice that the complex 8-D tangent vectors in coordinates  $z_1^k$  or  $z_2^k$  for complexified space-time surface and coordinates  $(z_1^k, z_2^k)$  for  $o$  have components

$$\frac{\partial o^i}{\partial z_1^k} \leftrightarrow (\delta_k^i, \frac{\partial z_2^i}{\partial z_1^k})$$

or

$$\left(\frac{\partial o^i}{\partial z_2^k}\right) \leftrightarrow \left(\frac{\partial z_1^i}{\partial z_2^k}, \delta_k^i\right) . \quad (4.11)$$

These vectors correspond to complexified octonions  $O_i$  given by

$$\delta_k^i E^k + \frac{\partial z_2^i}{\partial z_1^k} E^k E_4 , \quad (4.12)$$

where the unit octonions are given by  $(E_0, E_1, E_2, E_3) = (1, I_1, I_2, I_3)$  and  $(E_5, E_6, E_7, E_8) = (1, I_1, I_2, I_3)E_4$ . The vanishing of the associators stating that one has

4. One can calculate the partial derivatives  $\frac{\partial z_i^k}{\partial z_j^l}$  explicitly without solving the equations or the complex valued quaternionic components of  $RE(P) \equiv X = 0$  or  $IM(P) \equiv Y = 0$  (note that  $X$  and  $Y$  have for complex components labelled by  $X^i$  and  $Y^i$  respectively).

$$Y^i(z_1^k, z_2^l) = c \in R , \quad i = 1, \dots, 4 , \quad \text{associativity} ,$$

or

$$X^i(z_1^k, z_2^l) = c \in R , \quad i = 1, \dots, 4 , \quad \text{co-associativity} . \quad (4.13)$$

explicitly and check whether associativity holds true. The derivatives can be deduced from the constancy of  $Y$  or  $X$ .

5. For instance, if one has  $z_2^k$  as function of  $z_1^k$ , one obtains in the associative case

$$\begin{aligned} RE(Y)^i_k + IM(Y)^i_k \frac{\partial z_2^r}{\partial z_1^k} &= 0 \\ RE(Y)^i_k &\equiv \frac{\partial Y^i}{\partial z_1^k} , & IM(Y)^i_k &\equiv \frac{\partial Y^i}{\partial z_2^k} . \end{aligned} \quad (4.14)$$

In co-associative case one must consider normal vectors expressible in terms of  $Y^i$  so that  $X$  is replaced with  $Y$  in these equations.

This allows to solve the partial derivatives needed in associator conditions

$$\frac{\partial z_2^i}{\partial z_1^k} = [Im(Y)^{-1}]^i_r Re(Y)^r_k . \quad (4.15)$$

6. The vanishing conditions for the associators are however multilinear and one can multiply each factor by the matrix  $IM(P)$  without affecting the condition so that  $IM(P)^{-1}$  disappears and one obtains the conditions for vectors

$$\begin{aligned}
T_r^i T_s^j T_t^k f^{rstu} &= 0 \quad , \quad i, j, k \in \{1, \dots, 4\} \quad , \quad r, s, t, u \in \{1, \dots, 7\} \quad , \\
T^i &= IM(Y)^i_k E^k - RE(Y)^i_k E^k E_4 \quad .
\end{aligned}
\tag{4.16}$$

This form of conditions is computationally much more convenient.

How to solve these equations?

1. The antisymmetry of  $f^{rstu}$  with respect to the first two indices  $r, s$  leads one to ask whether one could have

$$T_r^i T_s^j T_t^k = 0 \tag{4.17}$$

for the 7 quaternionic triplets. This is guaranteed if one has either  $RE(Y)^i_k = \partial Y^i / \partial z_1^k = 0$  (coquaternionic part of  $T^i$ ) or  $IM(Y)^i_k = \partial Y^i / \partial z_2^k = 0$  (co-quaternionic part of  $T^i$ ) for *one* member in each triplet.

The study of the structure constants listed above shows that indices 1,2,3 are contained in all 7 triplets. Same holds for the indices appearing in any quaternionic triplet. Hence it is enough to require that three gradients  $RE(Y)^i_k = 0$  or  $IM(Y)^i_k = 0$   $k \in \{1, 2, 3\}$  vanish. This condition is obviously too strong since already single vanishing condition reduces the dimension of space-time variety to 3 in the generic case and it becomes trivially associative.

Octonionic automorphism group  $G_2$  gives additional basis with their own quaternion triplets and the general condition would be that 3 partial derivatives vanish for a triplet obtained from the basic triplet  $\{1, 2, 3\}$  by  $G_2$  transformation. It is not quite clear to me whether the  $G_2$  transformation can depend on position on space-time surface.

2. As noticed, the vanishing of all triplets is an un-necessarily strong condition. Already the vanishing of single gradient  $RE(Y)^i_k$  or  $IM(Y)^i_k$  reduces the dimension of the surface from 4 to 3 in the generic case. If one accepts that the dimension of associative surface is lower than 4 then single criticality condition is enough to obtain 3-D surface.

In the generic case associativity holds true only at the 4-D tangent spaces of 3-surfaces at the ends of CD (at light-like partonic orbits it holds true trivially in 4-D) and that the twistor lift of Kähler action determines the space-time surfaces in their interior.

In this case one can map only the boundaries of space-time surface by  $M^8 - H$  duality to  $H$ . The criticality at these 3-surfaces dictates the boundary conditions and provides a solution to infinite number of conditions stating the vanishing of SSA Noether charges at space-like boundaries. These space-time regions would correspond to the regions of space-time surfaces inside CDs identifiable as interaction regions, where Kähler action and volume term couple and dynamics depends on coupling constants.

The mappability of  $M^8$  dynamics to  $H$  dynamics in all space-time regions does not look feasible: the dynamics of octonionic polynomials involves no coupling constants whereas twistor lift of Kähler action involves couplings parameters. The dynamics would be non-associative in the geometric sense in the interior of CDs. Notice that also conformal field theories involve slight breaking of associativity and that octonions break associativity only slightly ( $a(bc) = -(ab)c$  for octonionic imaginary units). I have discussed the breaking of associativity from TGD viewpoint in [K13].

3. Twistor lift of Kähler action allows also space-time regions, which are minimal surfaces [L1] and for which the coupling between Kähler action and volume term vanishes. Preferred extremal property reduces to the existence of Hamilton-Jacobi structure as image of the quaternionic structure at the level of  $M^8$ . The dynamics is universal as also critical dynamics and independent of coupling constants so that  $M^8 - H$  duality makes sense for it. External particles arriving into CD via its boundaries would correspond to critical 4-surfaces: I have discussed their interpretation from the perspective of physics and biology in [L2].

4. One should be able to produce associativity without the reduction of dimension. One can indeed hope of obtaining 4-D associative surfaces by posing conditions on the coefficients of the polynomial  $P$  by requiring that one  $RE(Y)_k^i$  or  $IM(Y)_k^i$ ,  $i = i_1$  - call it just  $X_1$  - should vanish so that  $Y^i$  would be critical as function of  $z_1^k$  or  $z_2^k$ .

At  $X_1 = 0$  would have degenerate zero at the 4-surface. The decomposition of  $X_1$  to a product of monomial factors with root in extension of rationals would have one or more factors appearing at least twice. The associative 4-surfaces would be ramified. Also the physically interesting p-adic primes are conjectured to be ramified in the sense that their decomposition to primes of extension of rationals contains powers of primes of extension. The ramification of the monomial factors is nothing but ramification for polynomial primes in field of rationals in terms of polynomial primes in its extension.

This could lead to vanishing of say one triplet while keeping  $D = 4$ . This need not however give rise to associativity in which case also second  $RE(Y)_k^i$  or  $IM(Y)_k^i$ ,  $i = i_2$ , call it  $X_2$ , should vanish. The maximal number of  $X_i$  would be  $n_{max} = 3$ . The natural condition consistent with quantum criticality of TGD Universe would be that the variety is associative but maximally quantum critical and has therefore dimension  $D = 3$  or  $D = 4$ . Stronger condition allows only  $D = 4$ .

These  $n \leq 3$  additional conditions make the space-time surface analogous to a catastrophe with  $n \leq 3$  behavior variables in Thom's classification of 7 elementary catastrophes with less than 11 control variables [A1]. Thom's theory does not apply now since it has only one potential function  $V(x)$  (now  $n \leq 3$  corresponding to the critical coordinates  $Y^i$ !) as a function of behaviour variables and control variables). Also the number of non-vanishing coefficients in the polynomial having values in an extension of rationals and acting as control variables is unlimited. In quaternionic case the number of potential functions is indeed 1 but the number of control variables unlimited.

5. One should be able to understand the  $D = 3$  associative objects - say light-like 3-surfaces or 3-surfaces at the boundaries of CD - as 3-surfaces along which 4-D associative (co-associative) and non-associative (non-co-associative) surfaces are glued together.

Consider a product  $P$  of polynomials allowing 3-D surfaces as necessarily associative zero loci to which a small interaction polynomial vanishing at the boundaries of CD (proportional to  $o^n$ ,  $n > 1$ ) is added. Could  $P$  allow 4-D surface as a zero locus of real or imaginary part and containing the light-like 3-surfaces thanks to the presence of additional parameters coming from the interaction polynomial. Can one say that this small interaction polynomial would generate 4-D space-time in some sense? 4-D associative space-time regions would naturally emerge from the increasing algebraic complexity both via the increase of the degree of the polynomial and the increase of the dimension of the extension of rationals making it easier to satisfy the criticality conditions!

There are two regions to be considered: the interior and exterior of CD. Could associativity/co-associativity be possible outside CD but not inside CD so that one would indeed have free external particles entering to the non-associative interaction region. Why associativity conditions would be more difficult to satisfy inside CD? Certainly the space-likeness of  $M^4$  points with respect to the preferred origin of  $M^8$  in this region should be crucial since Minkowski norm appears in the expressions of  $RE(P)$  and  $IM(P)$ .

Do the calculations for the associative case generalize to the co-associative case?

1. Suppose that one has possibly associative surface having  $RE(P) = 0$ . One would have  $IM(P) = 0$  for dual space-time surface defining locally normal space of  $RE(P) = 0$  surface. This would transform the co-associativity conditions to associativity conditions and the preceding arguments should go through essentially as such.

Associative and co-associative surfaces would meet at singularity  $RE(P) = IM(P) = 0$ , which need not be point in Minkowskian signature (see  $P = o^2$  example in the Appendix) and can be even 4-D! This raises the possibility that the associative and co-associative surfaces defined by  $RE(P) = 0$  and  $IM(P) = 0$  meet along 3-D light-like orbits partonic surfaces or 3-D ends of space-time surfaces at the ends of CD.

2. If  $D = 3$  for associative surfaces are allowed besides  $D = 4$  as boundaries of 4-surfaces, one can ask why not allow  $D = 5$  for co-associative surfaces. It seems that they do not have a reasonable interpretation as a surface at which co-associative and non-co-associative 4-D space-time regions would meet. Or could they in some sense be geometric “co-boundaries” of 4-surfaces like branes in M-theory serve as co-boundaries of strings? Could this mean that 4-D space-time-surface is boundary of 5-D co-associative surface defining a TGD variant of brane with strings world sheets replaced with 4-D space-time surfaces?

What came as a surprise that  $P = o^2$  allows 5-D and 6-D surfaces as zero loci of  $RE(P)$  or  $IM(P)$  as shown in Appendix. The vanishing of the entire  $o^2$  gives 4-D interior or exterior of CD forced also by associativity/co-associativity and thus maximal quantum criticality. This is very probably due to the special properties of  $o^2$  as polynomial: in the generic case the zero loci should be 4-D.

This discussion can be repeated for complex/co-complex surfaces inside space-time surfaces associated with fermionic dynamics.

1. Associativity condition does not force string world sheets and partonic 2-surfaces but they could naturally correspond to commutative or co-commutative varieties inside associative/co-associative varieties.

In the generic case commutativity/co-commutativity allows only 1-D curves - naturally light-like fermionic world lines at the boundaries of partonic orbits and representing interacting point-like fermions inside CDs and used in the construction of twistor amplitudes [K15, K19]. There is coupling between Kähler part and volume parts of modified Dirac action inside CDs so that coupling constants are visible in the spinor dynamics and in dynamics of string world sheet.

2. At criticality one obtains 2-D commutative/co-commutative surfaces necessarily associated with external particles quantum critical in 4-D sense and allowing quaternionic structure. String world sheets would serve as correlates for bound state entanglement between fermions at their ends. Criticality condition would select string world sheets and partonic 2-surfaces from the slicing of space-time surface provided by quaternionic structure (having Hamilton-Jacobi structure as  $H$ -counterpart).

If associativity holds true and fixed  $M_c^2$  is contained in the tangent space of space-time surface, one can map the  $M^4$  projection of the space-time surface to a surface in  $M^4 \times CP_2$  so that the quaternionic tangent space at given point is mapped to  $CP_2$  point. One obtains 4-D surface in  $H = M^4 \times CP_2$ .

1. The condition that fixed  $M_c^2$  belongs to the tangent space of  $X_c^4$  is true in the sense that the coordinates  $z_2^k$  do not depend on  $z_1^1$  and  $z_1^2$  defining the coordinates of  $M_c^2$ . It is not clear whether this condition can be satisfied in the general case: octonionic polynomials are expected to imply this dependence un-avoidably.

The more general condition allows  $M_c^2$  to depend on position but assumes that  $M_c^2$ :s associated with different points integrate to a family 2-D surfaces defining a family of complexified string world sheets. In the similar manner the orthogonal complements  $E_c^2$  of  $M_c^2$  would integrate to a family of partonic 2-surfaces. At each point these two tangent spaces and their real projections would define a decomposition analogous to that define by light-like momentum vector and polarization vector orthogonal to it. This decomposition would define decomposition of quaternionic sub-spaces to complexified complex subspace and its co-complex normal space. The decomposition would correspond to Hamilton-Jacobi structure proposed to be central aspect of extremals [K12].

2. What is nice that this decomposition of  $M_c^4$  ( $M^4$ ) would (and of course should!) follow automatically from the octonionic decomposition. This decomposition is lower-dimensional analog to that of the complexified octonionic space induced by level sets of real octonionic polynomials but at lower level and extremely natural due to the inclusion hierarchy of classical number fields. Also  $M_c^2$  could have similar decomposition orthogonal complex curves by the



value sets of polynomials. The hierarchy of grids means the realization of the coordinate grid consisting of quaternionic, complex, and real curves for complexified coordinates  $o^k$  and their quaternionic and complex variants and is accompanied by corresponding real grids obtained by projecting to  $M^4$  and mapping to  $CP_2$ .

Thus these decompositions would be obtained from the octonionic polynomial decomposing it to real quaternionic and imaginary quaternionic parts first to get a grid of space-time surfaces as constant value sets of either real or imaginary part, doing the same for the non-constant quaternionic part of the octonionic polynomial to get similar grid of complexified 2-surfaces, and repeating this for the complexified complex octonionic part.

Unfortunately, I do not have computer power to check the associativity directly by symbolic calculation. I hope that the reader could perform the numerical calculations in non-trivial cases to this!

#### 4.2.6 General view about solutions to $RE(P) = 0$ and $IM(P) = 0$ conditions

The first challenge is to understand at general level the nature of  $RE(P) = 0$  and  $IM(P) = 0$  conditions. Appendix shows explicitly for  $P(o) = o^2$  that Minkowski signature gives rise to unexpected phenomena. In the following these phenomena are shown to be completely general but not quite what one obtains for  $P(o) = o^2$  having double root at origin.

1. Consider first the octonionic polynomials  $P(o)$  satisfying  $P(0) = 0$  restricted to the light-like boundary  $\delta M_+^8$  assignable to 8-D CD, where the octonionic norm of  $o$  vanishes.
  - (a)  $P(o)$  reduces along each light-ray of  $\delta M_+^8$  to the same real valued polynomial  $P(t)$  of a real variable  $t$  apart from a multiplicative unit  $E = (1 + in)/2$  satisfying  $E^2 = E$ . Here  $n$  is purely octonion-imaginary unit vector defining the direction of the light-ray.  $IM(P) = 0$  corresponds to quaternionicity. If the  $E^4$  ( $M^8 = M^4 \times E^4$ ) projection is vanishing, there is no additional condition. 4-D light-cones  $M_\pm^4$  are obtained as solutions of  $IM(P) = 0$ . Note that  $M_\pm^4$  can correspond to any quaternionic subspace. If the light-like ray has a non-vanishing projection to  $E^4$ , one must have  $P(t) = 0$ . The solutions form a collection of 6-spheres labelled by the roots  $t_n$  of  $P(t) = 0$ . 6-spheres are not associative.
  - (b)  $RE(PE) = 0$  corresponding to co-quaternionicity leads to  $P(t) = 0$  always and gives a collection of 6-spheres.
2. Suppose now that  $P(t)$  is shifted to  $P_1(t) = P(t) - c$ ,  $c$  a real number. Also now  $M_\pm^4$  is obtained as solutions to  $IM(P) = 0$ . For  $RE(P) = 0$  one obtains two conditions  $P(t) = 0$  and  $P(t - c) = 0$ . The common roots define a subset of 6-spheres which for special values of  $c$  is not empty.

The above discussion was limited to  $\delta M_+^8$  and light-likeness of its points played a central role. What about the interior of 8-D CD?

1. The natural expectation is that in the interior of CD one obtains a 4-D variety  $X^4$ . For  $IM(P) = 0$  the outcome would be union of  $X^4$  with  $M_\pm^4$  and the set of 6-spheres for  $IM(P) = 0$ . 4-D variety would intersect  $M_\pm^4$  in a discrete set of points and the 6-spheres along 2-D varieties  $X^2$ . The higher the degree of  $P$ , the larger the number of 6-spheres and these 2-varieties.
2. For  $RE(P) = 0$   $X^4$  would intersect the union of 6-spheres along 2-D varieties. What comes in mind that these 2-varieties correspond in  $H$  to partonic 2-surfaces defining light-like 3-surfaces at which the induced metric is degenerate.
3. One can consider also the situation in the complement of 8-D CD which corresponds to the complement of 4-D CD. One expects that  $RE(P) = 0$  condition is replaced with  $IM(P) = 0$  condition in the complement and  $RE(P) = IM(P) = 0$  holds true at the boundary of 4-D CD.

6-spheres and 4-D empty light-cones are special solutions of the conditions and clearly analogs of branes. Should one make the (reluctant-to-me) conclusion that they might be relevant for TGD at the level of  $M^8$ .

1. Could  $M^4_+$  (or CDs as 4-D objects) and 6-spheres integrate the space-time varieties inside different 4-D CDs to single connected structure with space-time varieties glued to the 6-spheres along 2-surfaces  $X^2$  perhaps identifiable as pre-images of partonic 2-surfaces and maybe string world sheets? Could the interactions between space-time varieties  $X^4_i$  assignable with different CDs be describable by regarding 6-spheres as bridges between  $X^4_i$  having only a discrete set of common points. Could one say that  $X^2_i$  interact via the 6-sphere somehow. Note however that 6-spheres are not dynamical.
2. One can also have Poincare transforms of 8-D CDs. Could the description of their interactions involve 4-D intersections of corresponding 6-spheres?
3. 6-spheres in  $IM(P) = 0$  case do not have image under  $M^8 - H$  correspondence. This does not seem to be possible for  $RE(P) = 0$  either: it is not possible to map the 2-D normal space to a unique  $CP_2$  point since there is 2-D continuum of quaternionic sub-spaces containing it.

### 4.3 $M^8 - H$ duality: objections and challenges

In the following I try to recall all objections against the reduction of classical physics to octonionic algebraic geometry and against the notion of  $M^8 - H$  duality and also invent some new counter arguments and challenges.

#### 4.3.1 Can one really assume distribution of $M^2(x)$ ?

Hamilton-Jacobi structure means that  $M^2(x)$  depends on position and  $M^2(x)$  should define an integrable distribution integrating to a 2-D surface. For cosmic string extremals this surface would be minimal surface so that the term “string world sheet” is appropriate. There are objections.

1. It seems that the coefficients of octonionic polynomials cannot contain information about string world sheet, and the only possible choice seems to be that string world sheets and partonic 2-surfaces parallel to it assigned with integrable distribution of orthogonal complements  $E^2(x)$  should be coded by quaternionic moduli. It should be possible to define quaternionic coordinates  $q_i$  decomposing to pairs of complex coordinates to each choice of  $M^2(x) \times E^2(x)$  decomposition of given  $M^4_0$ . Octonionic coordinates would be given by  $o = q_1 + q_2 I_4$  where  $q_i$  are associated with the same quaternionic moduli. The choice of Hamilton-Jacobi structure would not be ad hoc procedure anymore but part of the definition of solutions of field equations at the level of  $M^8$ .
2. It would be very nice if the quaternionic structure could be induced from a fixed structure defined for  $M^8_c$  once the choice of curvilinear  $M^4$  coordinates is made. Since Hamilton-Jacobi structure [K12] involves a choice of generalized Kähler form for  $M^4$  and since quaternionic structure means that there is full  $S^2$  of Kähler structures determined by quaternionic imaginary units (ordinary Kähler form for sub-space  $E^8 \subset M^8_c$ ) the natural proposal is that Hamilton-Jacobi structures is determined by a particular local choice of the Kähler form for  $M^4$  involving fixing of quaternionic imaginary unit fixing  $M^2(x) \subset M^4_0$  identifiable as point of  $S^2$ . This might relate closely also to the fixing of twistor structure, which indeed involves also self-dual Kähler form and a similar choice.
3. One can argue that it is not completely clear whether massless extremals (MEs) [K12] allow a general Hamilton-Jacobi structure. It is certainly true that if the light-like direction and orthogonal polarization direction are constant, MEs exist. It is clear that if the form of field equations is preserved and thus reduces to contractions of various tensors with second fundamental form one obtains only contractions of light-like vector with itself or polarization vector and these contractions vanish. For spatially varying directions one could argue that light-like direction codes for a direction of light-like momentum and that problems with local conservation laws expressed by field equations might emerge.

### 4.3.2 Can one assign to the tangent plane of $X^4 \subset M^8$ a unique $CP_2$ point when $M^2$ depends on position

One should show that the choice  $s(x) \in CP_2$  for a given distribution of  $M^2(x) \subset M^4(x)$  is unique in order to realize the  $M^8 - H$  correspondence as a map  $M^8 \rightarrow H$ . It would be even better if one had an analytic formula for  $s(x)$  using tangent space-data for  $X^4 \subset M^8$ .

1. If  $M^2(x) = M_0^2$  holds true but the tangent space  $M^4(x)$  depends on position, the assignment of  $CP_2$  point  $s(x)$  to the tangent space of  $X^4 \subset M^8$  is trivial. When  $M^4(x)$  is not constant, the situation is not so easy.
2. The space  $M^2(x) \subset M^4(x)$  satisfies also the constraint  $M^2(x) \subset M_0^4$  since quaternionic moduli are fixed. To avoid confusion notice that  $M^4(x)$  denotes tangent space of  $X^4$  and is different from  $M_0^4$  fixing the quaternionic moduli.
3.  $M^2(x)$  determines the local complex subspace and its completion to quaternionic tangent space  $M^4(x)$  determines a point  $s(x)$  of  $CP_2$ . The idea is that  $M_0^2$  defines a standard reference and that one should be able to map  $M^2(x)$  to  $M_0^2$  by  $G_2$  automorphism mapping also the  $s(x)$  to a unique point  $s_0(x) \in CP_2$  defining the  $CP_2$  point assignable to the point of  $X^4 \subset M^8$ .
4. One can assign to the point  $x$  quaternionic unit vector  $n(x)$  determining  $M^2(x)$  as the direction of the preferred imaginary unit. The  $G_2$  transformation must rotate  $n(x)$  to  $n_0$  defining  $M_0^2$  and acts on  $s$ .  $G_2$  transformation is not unique since  $u_1 g u_2$  has the same effect for  $u_i \in U(2)$  leaving invariant the point of  $CP_2$  for initial and final situation. Hence the equivalence classes of transformations should correspond to a point of 6-dimensional double coset space  $U(2) \backslash G_2 / U(2)$ . Intuitively it seems obvious that the  $s_0(x)$  is unique but proof is required.

### 4.3.3 What about the inverse of $M^8 - H$ duality?

$M^8 - H$  duality should have inverse in the critical regions of  $X^4 \subset M^8$ , where associativity conditions are satisfied. How could one construct the inverse of  $M^8 - H$  duality in these regions? One should map space-time points  $(m, s) \in M^4 \times CP_2$  to points  $(m, e) = (m, f(m, s)) \in M^8$ .  $M_0^4 \supset M_0^2$  parameterized by  $CP_2$  point can be chosen arbitrarily and one can require that it corresponds to some space-time point  $(m_0, s_0) \in H$ .  $CP_2$  point  $s(x)$  characterizes the quaternionic tangent space containing  $M^2(x)$  and can choose  $M_0^2$  to be  $M^2(x_0)$  for conveniently chosen  $x_0$ . Coordinates  $x$  can be used also for  $X^4 \subset M^8$ .

One obtains set of points  $(m, e) = (m(x), f(m(x), s(x))) \in M^8$  and a distribution of 4-D spaces of labelled by  $s(x)$ . This requires that the 4-D tangent space spanned by the gradients of  $m(x)$  and  $f(m(x), s(x))$  and characterized by  $s_1 \in CP_2$  for given  $M^2(x)$  by using the above procedure mapping the situation to that for  $M_0^2$  is same as the tangent space determined by  $s(x)$ :  $s(x) = s_1(x)$ . Also the associativity conditions should hold true. One should have a formula for  $s_1$  as function of tangent vectors of space-time surface in  $M^8$ . The ansatz based on algebraic geometry in  $M_c^8$  should be equivalent with this ansatz. The problem is that the ansatz leads to algebraic functions which cannot be found explicitly. It might be that in practice the correspondence is easy only in the direction  $M^8 \rightarrow H$ .

### 4.3.4 What one can say about twistor lift of $M^8 - H$ duality?

One can argue that the twistor spaces  $CP_1$  associated with  $M^4$  and  $E^4$  are in no way visible in the dynamics of octonion polynomials and in  $M^8 - H$  duality. Hence one could argue that they are not needed for any reasonable purpose. I cannot decide whether this is indeed the case. There I will consider the existence of twistor lift of the  $M^8$  and also the twistor lift  $M^8 - H$  duality in the space-time regions, where the tangent spaces satisfy the conditions for the existence of the duality as a map  $(m, e) \in M^8 \rightarrow (m, s) \in M^4 \times CP_2$  must be considered. This involves some non-trivial delicacies.

1. The twistor bundles of  $M_c^4$  and  $E_c^4$  would be simply  $M_c^4 \times CP_1$  and  $E_c^4 \times CP_1$ .  $CP_1 = S^2$  parameterizes direction vectors in 3-D Euclidian space having interpretation as unit quaternions so that this interpretation might make sense. The definition of twistor structure means a selection of a preferred quaternion unit and its representation as Kähler form so that these twistor bundles would have thus Kähler structure. Twistor lift replaces complex quaternionic surfaces with their twistor spaces with induced twistor structure.
2. In  $M^8$  the radii of the spheres  $CP_1$  associated with  $M^4$  and  $E^4$  would be most naturally identical whereas in  $M^4 \times CP_2$  they can be different since  $CP_2$  is moduli space. Is the value of the  $CP_2$  radius visible at all in the classical dynamics in the critical associative/co-associative space-time regions, where one has minimal surfaces. Criticality would suggest that besides coupling constants also parameters with dimension of length should disappear from the field equations. At least for the known extremals such as massless extremals,  $CP_2$  type extremals, and cosmic strings  $CP_2$  radius plays no role in the equations.  $CP_2$  radius comes however into play only in interaction regions defined by CDs since  $M^8 - H$  duality works only at the 3-D ends of space-time surface and at the partonic orbits. Therefore the different radii for the  $CP_1$  associated with  $CP_2$  and  $E^4$  cause no obvious problems.

Consider now the idea about twistor space as real part of octonionic twistor space regarded as quaternion-complex space.

1. One can regard  $CP_1 = S^2$  as the space of unit quaternions and it is natural to replace it with the 6-sphere  $S^6$  of octonionic imaginary units at the level of complexified octonions. The sphere of complexified (by  $i$ ) unit octonions is non-compact space since the norm is complex valued and this generalization looks neither attractive nor necessary since the projection to real numbers would eliminate the complex part.

The equations determining the twistor bundle of space-time surface can be indeed formulated as vanishing of the quaternionic imaginary part of  $S^6$  coordinates, and one obtains a reduction to quaternionic sphere  $S^2$  at space-time level.

If  $S^2$  is identified as sub-manifold  $S^2 \subset S^6$ , it can be chosen in very many manners (this is of course not necessary). The choices are parameterized by  $SO(7)/SO(3) \times SO(4)$  having dimension  $D = 12$ . This choice has no physical content visible at the level of  $H$ . Note that the Kähler structure determining Hamilton-Jaboci structure is fixed by the choice of preferred direction ( $M^2(x)$ ). If all these moduli are allowed, it seems that one has something resembling multiverse, the description at the level of  $M^8$  is deeper one and one must ask whether the space-time surfaces with different twistorial, octonionic, and quaternionic moduli can interact.

2. The resulting octonionic analog of twistor space should be mapped by  $M^8 - H$  corresponds to twistor space of space-time surface  $T(M^4) \times T(CP_2)$ . The radii of twistor spheres of  $T(M^4)$  and  $T(CP_2)$  are different and this should be also understood. It would seem that the radius of  $T(M^4)$  at  $H = M^4 \times CP_2$  side should correspond to that of  $T(M^4)$  at  $M^8$  side and thus to that of  $S^6$  as its geodesic sphere: Planck length is the natural proposal inspired by the physical interpretation of the twistor lift. The radius of  $T(CP_2)$  twistor sphere should correspond to that of  $CP_2$  and is about  $2^{12}$  Planck lengths.

Therefore the scale of  $CP_2$  would emerge as a scale of moduli space and does not seem to be present at the level of  $M^8$  as a separate scale.  $M^8$  level would correspond to what might be called Planckian realm analogous to that associated with strings before dynamical compactification which is now replaced with number theoretic compactification. The key question is what determines the ratio of the radii of  $CP_2$  scale to Planck for which favored value is  $2^{12}$  [K14]. Could quantum criticality determine this ratio?

## 5 Appendix: $o^2$ as a simple test case

Octonionic polynomial  $o^2$  serves as a simple testing case.  $o^2$  is not irreducible so that its properties might not be generic and it might be better to study polynomial of form  $P(o) = o + po^2$  instead.

Before continuing, some conventions are needed.

1. The convention is that in  $M^8 = M^1 \times E^7$   $E^7$  corresponds to purely imaginary complexified octonions in both octonionic sense and in the sense that they are proportional to  $i$ .  $M^1$  corresponds to octonions real in both senses. This corresponds to the signature  $(1, -1, -1, -1, \dots)$  for  $M^8$  metric obtained as restriction of complexified metric. For  $M^4 = M^1 \times E^3$  analogous conventions hold true.
2. Conjugation  $o = o_0 + o_k I_k \rightarrow \bar{o} \equiv o_0 - I_k o_k$  does not change the sign of  $i$ . Quaternions can be decomposed to real and imaginary parts and some notation is needed. The notation  $q = Re(q) + Im(q)$  seems to be the least clumsy one: here  $Im(q)$  is 3-vector.

The explicit expression in terms of quaternionic decomposition  $o = q_1 + q_2 I_4$  reads as

$$P(o) = o^2 = q_1^2 - q_2 \bar{q}_2 + (q_1 q_2 + q_2 \bar{q}_1) I_4 . \quad (5.1)$$

$o$  corresponds to complexified octonion and there are two options concerning the interpretation of  $M^4$  and  $E^4$ .  $M^4$  could correspond to quaternionic or co-quaternionic sub-space. I have assumed the first interpretation hitherto but actually the identification is not obvious. This two cases are different and must be treated both.

With these notations quaternionic inner product reads as

$$\begin{aligned} q_1 q_2 &= Re(q_1 q_2) + Im(q_1 q_2) , \\ Re(q_1 q_2) &= Re(q_1) Re(q_2) - Im(q_1) \cdot Im(q_2) , \\ Im(q_1 q_2) &= Re(q_1) Im(q_2) + Re(q_2) Im(q_1) + Im(q_1) \times Im(q_2) . \end{aligned} \quad (5.2)$$

Here  $a \cdot b$  denotes the inner product of 3-vectors and  $a \times b$  their cross product.

Note that one has real and imaginary parts of octonions as two quaternions and real and imaginary parts of quaternions. To avoid confusion, I will use  $RE$  and  $IM$  to denote the decomposition of octonions to quaternions and  $Re$  and  $Im$  for the decomposition of quaternions to real and imaginary parts.

One can express the  $RE(o^2)$  as

$$\begin{aligned} RE(o^2) &\equiv X \equiv q_1^2 - q_2 \bar{q}_2 , \\ Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) , \\ Im(X) &= Im(q_1^2) = 2Re(q_1) Im(q_1) . \end{aligned} \quad (5.3)$$

For  $IM(o^2)$  one has

$$\begin{aligned} IM(o^2) &\equiv Y = q_1 q_2 + q_2 \bar{q}_1 \\ Re(Y) &= 2Re(q_1) Re(q_2) , \\ Im(Y) &= Re(q_1) Im(q_2) - Re(q_2) Im(q_1) + Im(q_1) \times Im(q_2) . \end{aligned} \quad (5.4)$$

The essential point is that only  $RE(o^2)$  contains the complexified Euclidian norm  $q_2 \bar{q}_2$  which becomes Minkowskian of Euclidian norm depending on whether one identifies  $M^4$  as associative or co-associative surface in  $o_c^8$ .

## 5.1 Option I: $M^4$ is quaternionic

Consider first the condition  $RE(o^2) = 0$ . The condition decomposes to two conditions stating the vanishing of quaternionic real and imaginary parts:

$$Re(X) = Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{E^4}(q_2) = 0 ,$$

$$Im(X) = Im(q_1^2) = 2Re(q_1) Im(q_1) = 0 .$$

(5.5)

$Im(X) = 0$  is satisfied for  $Re(q_1) = 0$  or  $Im(q_1) = 0$  so that one has two options. This gives 1-D line in time direction of 3-D hyperplane as a solution for  $M^4$  factor.

$Re(X) = 0$  states  $N_{M^4}(q_1) = N_{E^4}(q_2)$ .  $q_2$  coordinate itself is free.  $N_{E^4}(q_2)$  is negative so that  $q_1$  must be space-like with respect to the  $N_{M^4}$  so that only the solution  $Re(q_1) = 0$  is possible. Therefore one has  $Re(q_1) = 0$  and  $N_{M^4}(q_1) = N_{E^4}(q_2)$ .

One can assign to each  $E^4$  point a section of hyperboloid with  $t = 0$  hyper-plane giving a sphere and the surface is 6-dimensional sphere bundle like variety! This is completely unexpected result and presumably is due to the additional accidental symmetries due to the octonionicity. Also the fact that  $o^2$  is not irreducible polynomial is a probably reason since for  $o$  the surface is 4-D. The addition of linear term is expected to remove the degeneracy.

Consider next the case  $IM(o^2) = 0$ . The conditions read now as

$$\begin{aligned} Re(Y) &= 2Re(q_1)Re(q_2) = 0 \quad , \\ Im(Y) &= Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) = 0 \quad . \end{aligned} \tag{5.6}$$

Since cross product is orthogonal to the factors  $Im(Y) = 0$  condition requires that  $Im(q_1)$  and  $Im(q_2)$  are parallel vectors:  $Im(q_1) = \lambda Im(q_2)$  and one has the condition  $Re(q_1) = \lambda Re(q_2)$  implying  $q_1 = \Lambda q_2$ . Therefore to each point of  $E^4$  is associated a line of  $M^4$ . The surface is 5-dimensional.

It is interesting to look what the situation is if both conditions are true so that one would have a singularity. In this case  $Re(q_1) = 0$  and  $Re(q_1) = \lambda Re(q_2)$  imply  $\lambda = 0$  so that  $q_1 = 0$  is obtained and the solution reduces to 4-D  $E^4$ , which would be co-associative.

## 5.2 Option II: $M^4$ is co-quaternionic

This case is obtained by the inspection of the previous calculation by looking what changes the identification of  $M^4$  as co-quaternionic factor means. Now  $q_1$  is Euclidian and  $q_2$  Minkowskian coordinate and  $q_2\bar{q}_2$  gives Minkowskian rather than Euclidian norm.

Consider first  $RE(o^2) = 0$  case.

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{M^4}(q_2) = 0 \quad , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \quad . \end{aligned} \tag{5.7}$$

$N_{M^4}(q_1) - N_{M^4}(q_2) = 0$  condition holds true now besides the condition  $Re(q_1) = 0$  or  $Im(q_1) = 0$  so that one has also now two options.

1. For  $Re(q_1) = 0$   $N_{M^4}(q_1)$  is non-positive and this must be the case for  $N_{M^4}(q_2)$  so that the *exterior* of the light-cone is selected. In this case the points of  $M^4$  with fixed  $N_{M^4}$  give rise to a 2-D intersection with  $Re(q_1) = 0$  hyper-plane that is sphere so that one has 6-D surface, kind of sphere bundle.
2. For  $Im(q_1) = 0$  Minkowski norm is positive and so must be corresponding norm in  $E^4$  so that in  $E^4$  surface has future light-cone as projection. This surface is 4-D. The emergence of future light-cone might provide justification for the emergence of CDs and zero energy ontology.

For  $IM(o^2)$  the discussion is same as in quaternionic case since norm does not appear in the equations.

At singularity both  $RE(o^2)$  and  $IM(o^2) = 0$  vanish. The condition  $q_1 = \Lambda q_2$  reduces to  $\Lambda = 0$  so that  $q_1 = 0$  is only allowed. This leaves only light-cone boundary under consideration.

The appearance of surfaces with dimension higher than 4 raises the question whether something is wrong. One could of course argue that associativity allows also lower than 4-D surfaces as associative surfaces and higher than 4-D surfaces as co-associative surfaces. At  $H$ -level one can say that one has 4-D surfaces. A good guess is that this behavior disappears when the linear term is absent and origin ceases to be a singularity.

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