

Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part II

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Abstract

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and $M^8 - H$ correspondence could generalize.

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k , $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i . In the generic case this gives 3-D surface.

In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H , and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i . Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces?

I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

Also a sketchy proposal for the description of interactions is discussed.

1. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety X^4 . The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices.

The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

2. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

3. The possibility of super octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

Scattering diagrams would be determined by points of space-time variety, which are in extension of rationals. In adelic physics the interpretation is as cognitive representations.

1. Cognitive representations are identified as sets of rational points for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8 - and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [?]
2. Some aspects related to homology charge (Kähler magnetic charge) and genus-generation correspondence are discussed. Both topological quantum numbers are central in the proposed model of elementary particles and it is interesting to see whether the picture is internally consistent and how algebraic variety property affects the situation. Also possible problems related to $h_{eff}/h = n$ hierarchy [adelicphysics realized in terms of n -fold coverings of space-time surfaces are discussed from this perspective.

1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

1.1 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.

Remark: Why not rational functions expressible as ratios $R = P_1/P_2$ of octonionic polynomials? It has become clear that one can develop physical arguments in favor of this option. The zero loci for $IM(P_i)$ would represent space-time varieties. Zero loci for $RE(P_1/P_2) = 0$ and $RE(P_1/P_2) = \infty$ would represent their interaction presumably realized as wormhole contacts connecting these varieties. In the sequel most considerations are for polynomials: the replacement of polynomials with rational functions does not introduce big differences and its discussed in the section “Gromov-Witten invariants, Riemann-Roch theorem, and Atiyah-Singer index theorem from TGD point of view”.

2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by CP_2 points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.
3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L1]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P) = 0$ can transform to $IM(P) = 0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.
4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adèle for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.
5. Also a connection with infinite primes is suggestive [K20]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be

interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to $M^8 - H$ duality. The strategy is simple: try to remember all previous objections against $M^8 - H$ duality and invent new ones since this is the best way to make real progress.

1.2 Topics to be discussed

1.2.1 Challenges of the octonionic algebraic geometry

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to octonionic algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction (see <http://tinyurl.com/ybuy1a2k>) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a $M^8 - H$ correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k , $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i . In the generic case this gives 3-D surface.

In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H , and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i . Space-time surface would be analogous to a polynomial with a multiple root.

Various components of octonion polynomial P of degree n are polynomials of same degree. Could criticality reduce to the degeneracy of roots for some component polynomials? Could P as a polynomial of real variable have degenerate roots?

The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A1] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.
4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

1.2.2 Description of interactions

Also a sketchy proposal for the description of interactions is discussed.

1. $IM(P_1P_2) = 0$ is satisfied for $IM(P_1) = 0$ and $IM(P_2) = 0$ since $IM(o_1o_2)$ is linear in $IM(o_i)$ and one obtains union of space-time varieties. $RE(P_1P_2) = 0$ cannot be satisfied in this way since $RE(o_1o_2)$ is not linear in $RE(o_i)$ so that the two varieties interact and this interaction could give rise to a wormhole contact connecting the two space-time varieties.
2. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety X^4 . The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices. S^6 and therefore also X^2 are doubly critical, S^6 is also singular surface.

The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

3. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

4. The possibility of super-octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward way to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic. Indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N} = 4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the appropriate extension of rationals.

Twistor Grassmannian construction of scattering amplitudes at the level of M^8 looks feasible. The amplitudes decompose to M^4 and CP_2 parts with similar structure with E^4 spin (electroweak isospin) replacing ordinary spin. The residue integrals over Grassmannians emerging from the conservation of M^4 and E^4 4-momenta would have same form and guarantee Yangian supersymmetry in both sectors. The counterpart for the product of delta functions associated with the “negative helicities” (weak isospins with negative sign) would be expressible as a delta function in the complement of $SU(3)$ Cartan algebra $U(1) \times U(1)$ by using exponential map.

1.2.3 About the analogs of Gromov-Witten invariants and branes in TGD

Gromov-Witten (G-W) invariants belong to the realm of quantum enumerative geometry briefly discussed in [L7]. They count numbers of points in the intersection of varieties (“branes”) with quantum intersection identified as the existence of “string world sheet(s)” intersecting the branes. Also octonionic geometry gives rise to brane like objects. G-W invariants are rational numbers but it is proposed that they could be integers in TGD framework.

Riemann-Roch theorem (RR) and its generalization Atiyah-Singer index theorem (AS) relate dimensions of various kinds of moduli spaces to topological invariants. The possible generalizations of RR and AS to octonionic framework and the implications of $M^8 - H$ duality for the possible generalizations are discussed. The adelic hierarchy of extensions of rationals and criticality conditions make the moduli spaces discrete so that one expects kind of particle in box type quantization selecting discrete points of moduli spaces about the dimension.

The discussion of RR as also the notion of infinite primes and infinite rationals as counterparts of zero energy states suggests that rational functions $R = P_1/P_2$ could be more appropriate than mere polynomials. The construction of space-time varieties would not be modified in essential way: one would have zero loci of $IM(P_i)$ identifiable as space-time sheets and zero- and ∞ -loci of $RE(P_1/P_2)$ naturally identifiable as wormhole contacts connecting the space-time sheets.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

2 Some challenges of octonionic algebraic geometry

Space-time surfaces in $H = M^4 \times CP_2$ identified as preferred extremals of twistor lift of Kähler action leads to rather detailed view about space-time surfaces as counterparts of particles. Does this picture follow from $X^4 \subset M^8$ picture and does this description bring in something genuinely new?

2.1 Could free many-particle states as zero loci for real or imaginary parts for products of octonionic polynomials

In algebraic geometry zeros for the products of polynomials give rise to disjoint varieties, which are disjoint unions of surfaces assignable to the individual surfaces and possibly having lower-dimensional intersections. For instance, for complex curves these intersections consist of points. For complex surfaces they are complex curves.

In the case of octonionic polynomial $P = RE(P) + IM(P)I_4$ (Re and Im are defined in quaternionic sense) one considers zeros of quaternionic polynomial $RE(P)$ or $IM(P)$.

1. Product polynomial $P = P_1P_2$ decomposes to

$$P = RE(P_1)RE(P_2) - IM(P_1)IM(P_2) + (RE(P_1)IM(P_1) + IM(P_1)RE(P_2))I_4 .$$

One can require vanishing of $RE(P)$ or $IM(P)$.

- (a) $IM(P)$ vanishes for

$$(RE(P_1) = 0, RE(P_2) = 0)$$

or

$$IM(P_1) = 0, IM(P_2) = 0) .$$

- (b) $RE(P)$ vanishes for

$$(RE(P_1) = 0, IM(P_2) = 0)$$

or

$$IM(P_1) = 0, RE(P_2) = 0) .$$

One could reduce the condition $RE(P) = 0$ to $IM(P) = 0$ by replacing $P = P_1 + P_2I_4$ with $P_2 - P_1I_4$. If this condition is satisfied for the factors, it is satisfied also for the product. The set of surfaces is a commutative and associative algebra for the condition $IM(P) = 0$. Note that the quaternionic moduli must be same for the members of product. If one has quantum superposition of quaternionic moduli, the many-particle state involves a superposition of products with same moduli.

As found, the condition $IM(P) = 0$ can transform to $RE(P) = 0$ at singularities having $RE(P) = 0, IM(P) = 0$.

2. The commutativity of the product means that the products are analogous to many-boson states. P^n would define an algebraic analog of Bose-Einstein condensate. Does this surface correspond to a state consisting of n identical particles or is this artefact of representation? As a limiting case of product of different polynomials it might have interpretation as genuine n -boson states.
3. The product of two polynomials defines a union of disjoint surfaces having discrete intersection in Euclidian signature. In Minkowskian signature the vanishing of $q\bar{q}$ (conjugation does not affect the sign of i and changes only the sign of $I_k!$) can give rise to 3-D light-cone. The non-commutativity of quaternions indeed can give rise to combinations of type $q\bar{q}$ in $RE(P)$ and $IM(P)$.

What about interactions?

1. Could one introduce interaction by simply adding a polynomial P_{int} to the product? This polynomial should be small outside interaction region. CD would define naturally interaction regions and the interaction terms should vanish at the boundaries of CD. This might be possible in Minkowskian signature, where $f(q^2)$ multiplying the interaction term might vanish at the boundary of CD: in Euclidian sector $q\bar{q} = 0$ would imply $q = 0$ but in Minkowskian sector it would give light-cone as solution. One should arrange $IM(P_{int})$ to be proportional to $q\bar{q}$ vanishing at the boundary of CD. Minkowskian signature could be crucial for the possibility to “turning interactions on”.
2. If the imaginary part of the interaction term is proportional $f_1(q^2)f_2((q - T)^2)$ (T is real and corresponds to the temporal distance between the tips of CD) with $f_i(0) = 0$, one could obtain asymptotic states reducing to disjoint unions of zero loci of P^i at the boundaries of CD. If the order of the perturbation terms is higher than the total order of polynomials P^i , one would obtain new roots and particle emission. Non-perturbative situation would correspond to a dramatic modification of the space-time surface as a zero locus of $IM(P)$. This picture would be M^8 counterpart for the reduction of preferred extremals to minimal surfaces analogous to geodesic lines near the boundaries of CD: preferred extremals reduce to extremals of both Kähler action and volume term in these regions [L1].

The singularities of scattering amplitudes at algebraic varieties of Grassmann manifolds are central in the twistor Grassmann program [B1, B4, B3]. Since twistor lift of TGD seems to be the correct manner to formulate classical TGD in H , one can wonder about the connection between space-time surfaces in M_c^8 and scattering amplitudes. Witten’s formulation of twistor amplitudes in terms of algebraic curves in CP_3 suggests a formulation of scattering amplitudes in terms of the 4-D algebraic varieties in M_c^8 as of course, also TGD itself [K15, K18]! Could the huge multi-local Yangian symmetries of twistor Grassmann amplitudes reduce to octonion analyticity.

2.2 Two alternative interpretations for the restriction to M^4 subspace of M_c^8

One must complexify M^8 so that one has complexified octonions M_c^8 . This means the addition of imaginary unit i commuting with octonionic imaginary units. The vanishing of real or imaginary part of octonionic polynomial in quaternionic sense ($o = q_1 + Jq_2$) defines the space-time surface. Octonionic polynomial itself is obtained from a real polynomial by algebraic continuation so that in information theoretic sense space-time is 1-D. The roots of this real polynomial fix the polynomial and therefore also space-time surface uniquely. 1-D line degenerates to a discrete set of points of an extension in information theoretic sense. In p-adic case one can allow p-adic pseudo constants and this gives a model for imagination.

The octonionic roots $x + iy$ of the real polynomial need not however be real. There are two options.

1. The original proposal in [L6, L8] was that the *projection* from M_c^8 to real M^4 (for which M^1 coordinate is real and E^3 coordinates are imaginary with respect to i !) defines the real space-time surface mappable by $M^8 - H$ duality to CP_2 .
2. An alternative option is that only the roots of the 4 vanishing polynomials as coordinates of M_c^4 belong to M^4 so that m^0 would be real root and m^k , $k = 1, \dots, 3$ imaginary with respect to $i \rightarrow -i$. M_c^8 coordinates would be invariant (“real”) under combined conjugation $i \rightarrow -i, I_k \rightarrow -I_k$. In the following I will speak about this property as *Minkowskian reality*. This could make sense.

What is remarkable that this could allow to identify CDs in very elegant manner: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

Consider now this in detail.

1. One can think of starting from one of the 4 vanishing conditions for the components of octonionic polynomial guaranteeing associativity. Assuming real roots and continuing one by one through all 4 conditions to obtain 4-D Minkowskian real regions. The time coordinate of M^4 coordinates is real and others purely imaginary with respect to $i \rightarrow -i$. If this region does not connect 3-D surface at the boundaries of real CD, one must make a new trial.

Cusp catastrophe determined as the zero locus of third order polynomial provides an example. There are regions with single real root, regions with two real roots (complex roots become real and identical) defining V-shaped boundary of cusp and regions with 3 real roots (the interior of the cusp).

2. The restriction of the octonionic polynomial to time axis m^0 identifiable as octonionic real axes is a real polynomial with algebraic coefficients. In this case the root and its conjugate with respect to i would define the same surface. One could say that the Galois group of the real polynomial characterizes the space-time surface although at points other than those at real axis (time axis) the Galois group can be different.

One could consider the local Galois group of the fourth quaternionic valued polynomial, say the part of quaternionic polynomial corresponding to real unit 1 when other components are required to vanish and give rise to coordinates in $M^8 \subset M_c^8$ - Minkowskian reality. The extension and its Galois group would depend on the point of space-time surface.

An interesting question is how strong conditions Minkowskian reality poses on the extension. Minkowskian reality seems to imply that E^3 roots are purely real so that for an octonionic polynomial obtained as a continuation of a *real* polynomial one expects that both root and complex conjugate should be allowed and that Galois group should contain Z_2 reflection $i \rightarrow -i$. Space-time surface would be at least 2-sheeted. Also the model for elementary particles forces this conclusion on physical grounds. Real as opposite to imagined would mean Minkowskian reality in mathematical sense. In the case of polynomials this description would make sense in p-adic case by allowing the coefficients of the polynomial be pseudo constants.

3. What data one could use to fix the space-time surface? Can one start directly from the real polynomial and regard its coefficients as WCW coordinates? This would be easy and elegant. Space-time surface could be determined as Minkowskian real roots of the octonionic polynomial. The condition that the space-time surface has ends at boundaries of given CD and the roots are not Minkowskian real outside it would pose conditions on the polynomial. If the coefficients of the polynomial are p-adic pseudo constants, this condition might be easy to satisfy.

The situation depends also on the coordinates used. For linear coordinates such as Minkowski coordinates Minkowskian reality looks natural. One can however consider also angle like coordinates representable only in terms of complex phases p-adically and coming as roots of unity and requiring complex extension: at H-side they are very natural. For instance, for CP_2 all coordinates would be naturally represented in this manner. For future light-cone one would have hyperbolic angle and 2 ordinary angles plus light-cone proper time which would be real and positive coordinate.

This picture conforms with the proposed picture. The point is that the time coordinate m^k can be real in the sense that they are linear combinations of complex roots, say powers for the roots of unity. $E_c^4 \subset M_c^8$ could be complex and contain also complex roots since $M^8 - H$ duality does not depend on whether tangent space is complex or not. Therefore would could have complex extensions.

2.3 Questions related to ZEO and CDs

Octonionic polynomials provide a promising approach to the understanding of ZEO and CDs. Light-like boundary of CD as also light-cone emerge naturally as zeros of octonionic polynomials. This does not yet give CDs and ZEO: one should have intersection of future and past directed light-cones. The intuitive picture is that one has a hierarchy of CDs and that also the space-time surfaces inside different CDs interact.

2.3.1 Some general observations about CDs

It is good to list some basic features of CDs, which appear as both 4-D and 8-D variants.

1. There are both 4-D and 8-D CDs defined as intersections of future and past directed light-cones with tips at say origin 0 at real point T at quaternionic or octonionic time axis. CDs can be contained inside each other. CDs form a fractal hierarchy with CDs within CDs: one can add smaller CDs with given CD in all possible ways and repeat the process for the sub-CDs. One can also allow overlapping CDs and one can ask whether CDs define the analog of covering of O so that one would have something analogous to a manifold.
2. The boundaries of two CDs (both 4-D and 8-D) can intersect along light-like ray. For 4-D CD the image of this ray in H is light-like ray in M^4 at boundary of CD. For 8-D CD the image is in general curved line and the question is whether the light-like curves representing fermion orbits at the orbits of partonic 2-surfaces could be images of these lines.
3. The 3-surfaces at the boundaries of the two 4-D CDs are expected to have a discrete intersection since $4 + 4$ conditions must be satisfied (say $RE(P_i^k) = 0$ for $i = 1, 2, k = 1, 4$). Along line octonionic coordinate reduces effectively to real coordinate since one has $E^2 = E$ for $E = (1 + in)/2$, n octonionic unit. The origins of CDs are shifted by a light-like vector kE so that the light-like coordinates differ by a shift: $t_2 = t_1 - k$. Therefore one has common zero for real polynomials $RE(P_1^k(t))$ and $RE(P_2^k(t - k))$.

Are these intersection points somehow special physically? Could they correspond to the ends of fermionic lines? Could it happen that the intersection is 1-D in some special cases? The example of o^2 suggest that this might be the case. Does 1-D intersection of 3-surfaces at boundaries of 8-D CDs make possible interaction between space-time surfaces assignable to separate CDs as suggested by the proposed TGD based twistorial construction of scattering amplitudes?

4. Both tips of CD define naturally an origin of quaternionic coordinates for $D = 4$ and the origin of octonionic coordinates for $D = 8$. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be along the real line (time axis) connecting the tips of CD. Only the translations in this specified direction are symmetries preserving the commutativity and associativity of the polynomial algebra.
5. One expects that also Lorentz boosts of 4-D CDs are relevant. Lorentz boosts leave second boundary of CD invariant and Lorentz transforms the other one. Same applies to 8-D CDs. Lorentz boosts define non-equivalent octonionic and quaternionic structures and it seems that one assume moduli spaces of them.

One can of course ask whether the still somewhat ad hoc notion of CD general enough. Should one generalize it to the analog of the polygonal diagram with light-like geodesic lines as its edges appearing in the twistor Grassmannian approach to scattering diagrams? Octonionic approach gives naturally the light-like boundaries assignable to CDs but leaves open the question whether more complex structures with light-like boundaries are possible. How do the space-time surfaces associated with different quaternionic structures of M^8 and with different positions of tips of CD interact?

2.3.2 The emergence of causal diamonds (CDs)

CDs are a key notion of zero energy ontology (ZEO). They should emerge from the number-theoretic dynamics somehow. How? In the following this question is approached from two different directions.

1. One can ask whether the emergence of CDs could be understood in terms of singularities of octonion polynomials located at the light-like boundaries of CDs. In Minkowskian case the complex norm $q\bar{q}_i$ is present in P . Could this allow to blow up the singular point to a 3-D boundary of light-cone and allow to understand the emergence of causal diamonds (CDs) crucial in ZEO. This question will be considered below.

2. These arguments were developed before the realization that the Minkowskian reality condition discussed in the previous section is natural for the space-time surfaces as roots of the 4 polynomials defining real or imaginary part of octonionic polynomial in quaternionic sense and giving M^4 point as a solution. Minkowskian reality can hold only in some regions of M^4 and an attractive conjecture is that it fails outside CD. CD would be a prediction of number theoretical dynamics and have counterpart also at the level of H .

Consider now the second approach in more detail. The study of the special properties for zero loci of general polynomial $P(o)$ at light-rays of O indeed demonstrated that both 8-D and 4-D light-cones and their complements emerge naturally, and that the M^4 projections of these light-cones and even of their boundaries are 4-D future - or past directed light-cones. What one should understand is how CDs as their intersections, and therefore ZEO, emerge.

1. One manner to obtain CDs naturally is that the polynomials are sums $P(t) = \sum_k P_k(o)$ of products of form $P_k(o) = P_{1,k}(o)P_{2,k}(o - T)$, where T is real octonion defining the time coordinate. Single product of this kind gives two disjoint 4-varieties inside future and past directed light-cones $M^4_+(0)$ and $M^4_-(T)$ for either $RE(P) = 0$ (or $IM(P) = 0$) condition. The complements of these cones correspond to $IM(P) = 0$ (or $RE(P) = 0$) condition.
2. If one has nontrivial sum over the products, one obtains a connected 4-variety due the interaction terms. One has also as special solutions M^4_\pm and the 6-spheres associated with the zeros $P(t)$ or equivalently $P_1(t_1) \equiv P(t)$, $t_1 = T - t$ vanishing at the upper tip of CD. The causal diamond $M^4_+(0) \cap M^4_-(T)$ belongs to the intersection.

Remark: Also the union $M^4_-(0) \cup M^4_+(T)$ past and future directed light-cones belongs to the intersection but the latter is not considered in the proposed physical interpretation.

3. The time values defined by the roots t_n of $P(t)$ define a sequence of 6-spheres intersecting 4-D CD along 3-balls at times t_n . These time slices of CD must be physically somehow special. Space-time variety intersects 6-spheres along 2-varieties X_n^2 at times t_n . The varieties X_n^2 are perhaps identifiable as 2-D interaction vertices, pre-images of corresponding vertices in H at which the light-like orbits of partonic 2-surfaces arriving from the opposite boundaries of CD meet.

The expectation is that in H one has a generalized Feynman diagram with interaction vertices at times t_n . The higher the evolutionary level in algebraic sense is, the higher the degree of the polynomial $P(t)$, the number of t_n , and more complex the algebraic numbers t_n . $P(t)$ would be coded by the values of interaction times t_n . If their number is measurable, it would provide important information about the extension of rationals defining the evolutionary level. One can also hope of measuring t_n with some accuracy! Octonionic dynamics would solve the roots of a polynomial! This would give a direct connection with adelic physics [K8] [L9].

Remark: Could corresponding construction for higher algebras obtained by Cayley-Dickson construction solve the “roots” of polynomials with larger number of variables? Or could Cartesian product of octonionic spaces perhaps needed to describe interactions of CDs with arbitrary positions of tips lead to this?

4. Above I have considered only the interiors of light-cones. Also their complements are possible. The natural possibility is that varieties with $RE(P) = 0$ and $IM(P) = 0$ are glued at the boundary of CD, where $RE(P) = IM(P) = 0$ is satisfied. The complement should contain the external (free) particles, and the natural expectation is that in this region the associativity/co-associativity conditions can be satisfied.
5. The 4-varieties representing external particles would be glued at boundaries of CD to the interacting non-associative solution in the complement of CD. The interaction terms should be non-vanishing only inside CD so that in the exterior one would have just product $P(o) = P_{1,k_0}(o)P_{2,k_0}(o - T)$ giving rise to a disjoint union of associative varieties representing external particles. In the interior one could have interaction terms proportional to say $t^2(T - t)^2$ vanishing at the boundaries of CD in accordance with the idea that the interactions are switched on slowly. These terms would spoil the associativity.

Remark: One can also consider sums of the products $\prod_k P_k(o - T_k)$ of n polynomials and this gives a sequence of curves intersecting at their tips. It seems that something else is required to make the picture physical.

2.4 About singularities of octonionic algebraic varieties

In Minkowskian signature the notion of singularity for octonionic polynomials involves new aspects as the study of o^2 singular at origin shows (see Appendix). The region in which $RE(o^2) = 0, IM(o^2) = 0$ holds true is 4-D rather than a discrete set of points as one would naïvely expect.

1. At singularity the local dimension of the algebraic variety is reduced. For instance, double cone of 3-space has origin as singular point where it becomes 0-dimensional. A more general example is local pinch in which cylinder becomes infinitely thin at some point. This kind of pinching could occur for fibrations as the fiber contracts to a lower-dimensional space along a sub-variety of the base space.

A very simple analogy for this kind of singularity is the singularity of $P(x, y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x, y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

The signature of the singularity of algebraic variety determined by the conditions $P^i(z^j) = 0$ is the reduction of the maximal rank r for the matrix formed by the partial derivatives $P_j^i \equiv \partial IM(P)^i / \partial z^j$ ("RE" could replace "IM"). Rank corresponds to the largest dimension of the minor of P_j^i with non-vanishing determinant. Determinant vanishes when two rows of the minor are proportional to each other meaning that two tangent vectors become linearly dependent. When the rank is reduced by Δr , one has $r = r_{max} - \Delta r$ and the local dimension is locally reduced by Δr . One has hierarchy of singularities within singularities.

The conditions that all independent minors of the P_j^i have reduced rank gives additional constraints and define a sub-variety of the algebraic variety. Note that the dimension of the singularity corresponds to $d_s = \Delta r$ in the sense that the dimension of tangent space at singularity is effectively d_s .

2. In the recent case there are 4 polynomials and 4 complex variables so that $IM(P)_j^i$ is 4×4 -matrix. Its rank r can have values in $r = 1, 2, 3, 2, 4$. One can use Thom's catastrophe theory as a guideline. Catastrophe decomposes to pieces of various dimensions characterized by the reduction of the rank of the matrix defined by the second derivatives $V_{ij} = \partial_i \partial_j V$ of the potential function defining the catastrophe. For instance, for cusp catastrophe with $V(x, a, b) = x^4 + ax^2 + bx$ one has V-shaped region in (a, b) plane with maximal reduction of rank to $r = 0$ ($\partial_x^2 V = 0$) at the tip $(a, b) = 0$ at reduction to $r = 1$ at the sides of V , where two roots of $\partial_x V = 4x^3 + 2ax + b = 0$ co-incide requiring that the discriminant of this equation vanishes.
3. In the recent case $IM(P)$ takes the role of complex quaternion valued potential function and the 4 coordinates $z_1^{(k)}$ that of behavior variable x for cusp and $z_2^{(k)}$ that of control parameters (a, b) . The reduction of the rank of $n \times n$ matrix by Δr means that there are r linearly independent rows in the matrix. These give Δr additional conditions besides $IM(P) = 0$ so that the sub-variety along which the singularity takes places as dimension r . One can say that the r -dimensional tangent spaces integrate to the singular variety of dimension r .

The analogy with branes would be realized as a hierarchical structure of singularities of the spacetime surfaces. This hierarchy of singularities would realize space-time correlates for quantum criticality, which is basic principle of quantum TGD. For instance, the reduction by 3-units would correspond to strings - say at the ends of CD and along the partonic orbits (fermion lines), and maximal reduction might correspond to discrete points - say the ends of fermion lines at partonic 2-surfaces. Also isolated intersection points can be regarded as singularities and are stably present but it does not make sense to add fermions to these points so that cognitive representations are not possible.

4. Note that also the associativity - and commutativity conditions already discuss involved the gradients of $IM(P)^i$ and $RE(P)^i$, which would suggests that these regions can be interpreted as singularities for which the dimension is not lowered by on unit since the vanishing conditions hold true identically by criticality.

There are two cases to be considered. The usual Euclidian case in which pinch reducing the dimension and the Minkowskian case in which metric dimension is reduced locally.

Consider first the Euclidian case.

1. In Euclidian case it is difficult to tell whether all values of Δr are possible since octonion analyticity poses strong conditions on the singularities. The pinch could correspond to the singularity of the covering associated with the space-time surface defined by Galois group for the covering associated with $h_{eff}/h = n$ identifiable as the dimension of the extension [L4]. Therefore there would be very close connection between the extensions of rationals defining the Galois group and the extension of polynomial ring of 8 complex variables z_i^k , $i = 1, 2$, $k = 1, \dots, 4$ by algebraic functions. At the pinch, which would be algebraic point, the Galois group would have subgroup leaving the coordinates of the point invariant and some sheets of the covering defining roots would co-incide.
2. A very simple analogy for this kind of singularity is the singularity of $P(x, y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x, y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.
3. Quaternion structure predicts the slicing of M^4 by string world sheets inducing that of space-time surfaces. One must ask whether singular space-time sheets emerge already for the slicing of M^4 by string world sheets. String world sheets could be considered as candidates for $\Delta r = 2$ singularities of this kind. The physical intuition strongly suggests that there indeed physically preferred string world sheets and identification as $\Delta r = 2$ singularities of Euclidian type is attractive. Partonic 2-surfaces are also candidates in this respect. Could some sheets of the $h_{eff}/h = n$ covering co-incide at string world sheets?

Consider next the Minkowskian case. At the level of H the rank of the induced metric is reduced. This reduction need not be same as that for the matrix P_j^i and it is of course not obvious that the partonic orbit allows description as a singularity of algebraic variety.

1. Could the matrix P_j^i take a role analogous to the dual of induced metric and one might hope that the change of the sign for P_j^i for a fixed polynomial at singular surface could be analogous to the change of the sign of $\sqrt{g_4}$ so that the idea about algebraization of this singularity at level of M^8 might make sense. The information about metric could come from the fact that $IM(P)$ depends on complex valued quaternion norm reducing to Minkowskian metric in Minkowskian sub-space.
2. The condition for the reduction of rank from its maximal value of $r = 4$ to $r = 3$ occurs if one has $det(P) = 0$, which defines co-dimension 1 surface as a sub-variety of space-time surface. The interpretation as co-incidence of two roots should make sense if $IM(P) = 0$. Root pairs would now correspond now to the points at different sides of the singular 3-surface.

Minkowskian singularity cannot be identified as the 3-D space-like boundary of many-sheeted space-time surface located at the boundary of CD (induced metric is space-like).

Could this sub-variety be identified as partonic orbit, the common boundary of the Euclidian and Minkowskian regions? This would require that associative region transforms to co-associative one here. $IM(P) = 0$ condition can transform to $RE(P) = 0$ condition if one has $P = 0$ at this surface. Minkowskian variant of point singularity (P_j^i vanishes) would explode it to a light-like partonic orbit.

What does this imply about the rank of singularity? The condition $IM(P) = RE(P) = 0$ does not reduce the rank if P is linear polynomial and one could consider a hierarchy of reductions of rank. Since $q\bar{q}$ vanishes in Minkowskian sub-space at light-cone boundary

rather than at point $q = 0$ only, there are reasons to expect that it appears in P and reduces the rank by $\Delta r = 4$ (see Appendix for the discussion of o^2 case). The rank of the induced 4-metric is however reduced only by $\Delta r = 1$ at partonic orbit. If the complexified complex norm $z\bar{z}$, $z = z_1 + z_2 I_2$ can take the role of $q\bar{q}$, one has $\Delta r = 2$.

3. The reduction of rank to $r = 2$ would give rise to 2-surfaces, which are at the boundaries of 3-D singularities. If partonic orbits correspond to $\Delta r = 1$ singularities one could identify them as partonic 2-surfaces at the ends partonic orbits.

Could the singularity at partonic 2-surface correspond to the reduction of the rank of the induced metric by 2 units? This is impossible in strict sense since there is only one light-like direction in signature $(1, -1, -1, -1)$. Partonic 2-surface singularity would however correspond to a corner for both Euclidian and Minkowskian regions at which the metrically 2-D but topologically 3-D partonic orbit meets the the space-like 3-surface along the light-like boundary of CD. Also the radial direction for space-like 3-surface could become light-like at partonic 2-surface if the CP_2 coordinates have vanishing gradient with respect to the light-like radial coordinate r_M at the partonic 2-surface. In this sense the rank could be reduced by 2 units. The situation is analogous to that for fold singularity $y^2 - x = 0$.

String world sheets cannot be subsets of $r = 3$ singularities, which suggests different interpretation for partonic 2-surfaces and string world sheets.

What could this different interpretation be?

1. Perhaps the most convincing interpretation of string world sheets/partonic 2-surfaces has been already discussed (this interpretation would generalize to associative space-time surfaces). They could be commutative/co-commutative (here permutation might be allowed!) sub-manifolds of associative regions of the space-time surface allowing quaternionic tangent spaces so that the notions of commutative and co-commutative make sense. The criticality conditions are satisfied without the reduction of dimension from $d = 2$ to $d = 1$. In non-associative regions string world sheets would reduce to 1-D curves. This would happen at the boundaries of partonic orbits and 3-surfaces at the ends of space-time surface and only the ends of strings at partonic orbits carrying fermion number would be needed to determine twistorial scattering amplitudes [K15, K18].
2. I have also considered an interpretation in terms of singularities of space-time surfaces represented as a sections of their own twistor bundle. Self-intersections of the space-time surface would correspond to 2-D surfaces in this case [L4] and perhaps identifiable as string world sheets. The interpretation mentioned above would be in terms of Euclidian singularities. If this is true, the question is only about whether these two interpretations are consistent with each other.

If I were forced to draw conclusion on basis of these notices, it would be that only $r = 4$ Minkowskian singularities could be interesting and at them $RE(P) = 0$ regions could be transformed to $IM(P) = 0$ regions. Furthermore, the reduction of rank for the induced metric cannot be equal to the reduction of the rank for P_j^i .

2.5 The decomposition of space-time surface to Euclidian and Minkowskian regions in octonionic description

The unavoidable outcome of H picture is the decomposition of space-time surface to regions with Minkowskian or Euclidian signature of the induced metric. These regions are bounded by 3-D regions at which the signature of the induced metric is $(0, -1, -1, -1)$ due to the vanishing of the determinant of the induced metric. The boundary is naturally the light-like orbit of partonic 2-surface although one can consider also the possibility that these regions have boundaries intersecting along light-like curves defining boundaries of string world sheets. A more detailed view inspired by the study of extremals is following.

1. Let us assume that the above picture about decomposition of space-time surfaces in H to two kinds regions takes place. The regions where the dynamicis universal minimal surface

dynamics have associative pre-image in M^8 . The regions where Kähler action and volume term couple the associative pre-image in M^8 exists only at the 3-D boundary regions and M^8 dynamics determines the boundary conditions for H dynamics, which by holography is enough.

2. In the space-time regions having associative pre-image in M^8 one has a fibration of X^4 with with partonic surface as a local base and string world sheet as local fiber. In the interior of space-time region there are no singularities but at the boundary 2-D string world sheets becomes metrically 1-D as 1-D string boundary reduces metrically to 0-D structure analogous to a point. This reduction of dimension would be metric, but not topological.

The singularity for plane curve $P(x, y) = y^2 - x^3 = 0$ at origin illustrates the difference between Minkowskian and Euclidian singularity. One has $(\partial_x P, \partial_y P) = (-3x^2, 2y)$ vanishing at origin so that $\Delta r = 1$ singularity is in question and the dimension of singular manifold is indeed $r = 0$. From $y = \pm x^{3/2}$, $x \geq 0$. The induced metric $g_{xx} = 1 + (dy/dx)^2 = 1 + (9/4)x$, $x \geq 0$ is however non-singular at origin.

3. If the Euclidian region with pre-image corresponds to a deformation of wormhole contact, the identification as image of a co-associative space-time region in M^8 is natural so that normal space is associative and contains also the preferred $M^2(x)$. In Minkowskian regions the identification as image of associative space-time region in M^8 is natural.

What can one say about the relationship of the M^8 counterparts of neighboring Minkowskian and Euclidian regions?

1. Do these regions intersect along light-like 3-surfaces, 1-D light-like curve (orbit of fermion) or is the intersection discrete set of points possibly assignable to the partonic 2-surface at the boundaries of CD? The M^4 projections of the inverse image of the light-like partonic orbit should co-incide but E^4 projections need not do so. They could be however mappable to the same partonic two surface in $M^8 - H$ correspondence or the images could have at least have light-like curve as common.
2. It seems impossible for the space-time surfaces determined as zeros of octonionic polynomials to have boundaries. Rather, it seems that the boundary must be between Minkowskian and Euclidian regions of the space-time surface determined by the same octonionic polynomial. At the boundary also associative region would transform to co-associative region suggesting that $IM(P) = RE(P) = 0$ holds allowing to change the condition from $IM(P) = 0$ to $RE(P) = 0$.

Consider now in more detail whether this view can be realized.

1. In $H = M^4 \times CP_2$ the boundary between the Minkowskian and Euclidian space-time regions - light-like partonic 3-surface - is a singularity possible only in Minkowskian signature. Space-time surface X^4 at the boundary is effectively 3-D since one has $\sqrt{g_4} = 0$ meaning that tangent space is effectively 3-D. The 3-D boundary itself is metrically 2-D and this gives rise to the extended conformal invariance defining crucial distinction between TGD and super string models.
2. The singularities of $P(o)$ for o identified as linear coordinate of M_c^8 were already considered. The singularities correspond to the boundaries of light-cone and the emergence of CDs can be understood. Could also the light-like orbits of partonic 2-surfaces be understood in the same manner? Does the pre-image of this singularity in M^8 emerge as a singularity of an algebraic variety determined by the vanishing of $IM(P)$ for the octonionic polynomial?

What is common is that the rank of the induced metric by one unit also now. Now one has however also $\det(g_4) = 0$. The singularities correspond to curved light-like 3-surfaces inside space-time surfaces rather than light-like surfaces in M^8 : induced metric matters rather than M^4 metric.

3. Could also these regions correspond to singularities of octonionic polynomials at which $P(o) = 0$ is satisfied and associative region transforms to a co-associative region? For $M^2(x) = M_0^2$

this is impossible. Partonic 2-surfaces are planes E^2 now. One should have closed partonic 2-surfaces.

Could the allowance of quaternionic structures with slicing of X^4 by string world sheets and partonic 2-surfaces help? If one has slicing of string world sheets by dual light-like curves corresponding to light-like coordinates u and v , this slicing gives also rise to a slicing of light-like 3-surfaces and dual light-like coordinate. The pair (u, v) in fact defines the analog of z and \bar{z} in hypercomplex case. Could the singularity of $P(o)$ using the quaternionic coordinates defined by (u, v) and coordinates of partonic 2-surface allow to identify light-like partonic orbits with $\det(g_4) = 0$ as a generalization of light-cone boundaries in M^4 ?

The decomposition $M_0^4 = M_x^2 \times E^2(x)$ associated with quaternionic structure is independent of E^4 . In the other hand, tangent space of space-time surface at point decomposes $M^2(x) \times E_T^2(x)$, where $E_T^2(x)$ is in general different from $E^2(x)$. Is this enough to obtain partonic 2-surfaces as singularities with $RE(P) = IM(P) = 0$?

The question whether the boundaries between Minkowskian and Euclidian can correspond to singular regions at which $P(o)$ vanishes and the surface $RE(P) = 0$ transforms to $IM(P) = 0$ surface remains open. What remains poorly understood is the role of the induced metric. My hope is that with a further work the picture could be made more detailed.

2.6 About rational points of space-time surface

What one can say about rational points of space-time surface?

1. An important special case corresponds to a generalization of so called rational surfaces for which a parametric representation in terms of 4 complex coordinates t^k exists such that o_1^k are *rational* functions of t^k . The singularities for 2-complex dimensional surfaces in C^3 or equivalently CP_3 are classified by Du Val [A3, A4] (see <http://tinyurl.com/ydz93hle>).
2. In [L4] [K10] I considered possible singularities of the twistor bundle. These would correspond typically 2-D self-intersections of the imbedding of space-time surfaces as 4-D base space of 6-D twistor bundle with sphere as a fiber. They could relate to string world sheets and partonic 2-surfaces and - as already found - are different from singularities at the level of M_c^8 . The singularities of string world sheets and partonic 2-surfaces as hyper-complex and co-complex surfaces consist of points and could relate to the singularities at octonionic level.

As already mentioned, Bombieri-Lang conjecture (see <http://tinyurl.com/y887yn5b>) states that, for any variety X of general type over a number field k , the set of k -rational points of X is not Zariski dense (see <http://tinyurl.com/jm9fh74>) in X . Even more, the k -rational points are contained in a *finite* union of lower-dimensional sub-varieties of X .

This conjecture is highly interesting from TGD point of view if one believes in $M^8 - H$ duality. Space-time surfaces $X^4 \subset M_c^8$ can be seen as $M^8 = M^4 \times E^4$ projections of zero loci for real or imaginary parts of octonionic polynomials in o . In complex sense they reduce to $M^4 \times E^4$ projections of algebraic co-dimension 4 surfaces in C^8 . If Bombieri-Lang conjectures makes sense in this context, it would state that for a space-time surface $X^4 \subset M^8$ of general type the rational points are contained in a *finite* union of lower-dimensional sub-varieties. Also the conjecture of Vojta (see <http://tinyurl.com/y9sttuu4>) stating that varieties of general type cannot be potentially dense is known to be true for curves and support this general vision.

Could the finite union of sub-varieties correspond to string world sheets, partonic 2-surfaces, and their light-like orbits define singularities? But why just singular sub-varieties would be cognitively simple and have small Kodaira dimension d_K allowing large number of rational points? In the case of partonic orbits one might understand this as a reduction of metric dimension. The orbit is effectively 2-dimensional partonic surface metrically and for the genera $g = 0, 1$ rational points are dense. For string world sheets with handle number smaller than 2 the situation is same.

The proposed realizations of associativity and commutativity provide additional support for this picture. Criticality guaranteeing associativity/commutativity would select preferred space-time surfaces as also string world sheets and partonic 2-surfaces.

Concluding, the general wisdom of algebraic geometry conforms with SH and with the vision about the localization of cognitive representations at 2-surfaces. There are of many possible options for detailed interpretation and certainly the above sketch cannot be correct at the level of details.

2.7 About $h_{eff}/h = n$ as the number of sheets of Galois covering

The following considerations were motivated by the observation of a very stupid mistake that I have made repeatedly in some articles about TGD. Planck constant $h_{eff}/h = n$ corresponds naturally to the number of sheets of the covering space defined by the space-time surface.

I have however claimed that one has $n = ord(G)$, where $ord(G)$ is the order of the Galois group G associated with the extension of rationals assignable to the sector of “world of classical worlds” (WCW) and the dynamics of the space-time surface (what this means will be considered below).

This claim of course cannot be true since the generic point of extension G has some subgroup H leaving it invariant and one has $n = ord(G)/ord(H)$ dividing $ord(G)$. Equality holds true only for Abelian extensions with cyclic G . For singular points isotropy group is $H_1 \supset H$ so that $ord(H_1)/ord(H)$ sheets of the covering touch each other. I do not know how I have ended up to a conclusion, which is so obviously wrong, and how I have managed for so long to not notice my blunder.

This observation forced me to consider more precisely what the idea about Galois group acting as a number theoretic symmetry group really means at space-time level and it turned out that $M^8 - H$ correspondence [L6] (see <http://tinyurl.com/yd43o2n2>) gives a precise meaning for this idea.

Consider first the action of Galois group (see <http://tinyurl.com/y8grabt2> and <http://tinyurl.com/yzze5psx>).

1. The action of Galois group leaves invariant the number theoretic norm characterizing the extension. The generic orbit of Galois group can be regarded as a discrete coset space G/H , $H \subset G$. The action of Galois group is transitive for irreducible polynomials so that any two points at the orbit are G -related. For the singular points the isotropy group is larger than for generic points and the orbit is G/H_1 , $H_1 \supset H$ so that the number of points of the orbit divides n . Since rationals remain invariant under G , the orbit of any rational point contains only single point. The orbit of a point in the complement of rationals under G is analogous to an orbit of a point of sphere under discrete subgroup of $SO(3)$.

$n = ord(G)/ord(H)$ divides the order $ord(G)$ of Galois group G . The largest possible Galois group for n -D algebraic extension is permutation group S_n . A theorem of Frobenius states that this can be achieved for $n = p$, p prime if there is only single pair of complex roots (see <http://tinyurl.com/y8grabt2>). Prime-dimensional extensions with $h_{eff}/h = p$ would have maximal number theoretical symmetries and could be very special physically: p-adic physics again!

2. The action of G on a point of space-time surface with embedding space coordinates in n -D extension of rationals gives rise to an orbit containing n points except when the isotropy group leaving the point is larger than for a generic point. One therefore obtains singular covering with the sheets of the covering touching each other at singular points. Rational points are maximally singular points at which all sheets of the covering touch each other.
3. At QFT limit of TGD the n dynamically identical sheets of covering are effectively replaced with single one and this effectively replaces h with $h_{eff} = n \times h$ in the exponent of action (Planck constant is still the familiar h at the fundamental level). n is naturally the dimension of the extension and thus satisfies $n \leq ord(G)$. $n = ord(G)$ is satisfied only if G is cyclic group.

The challenge is to define what space-time surface as Galois covering does really mean!

1. The surface considered can be partonic 2-surface, string world sheet, space-like 3-surface at the boundary of CD, light-like orbit of partonic 2-surface, or space-time surface. What one actually has is only the data given by these discrete points having embedding space coordinates in a given extension of rationals. One considers an extension of rationals determined by irreducible polynomial P but in p-adic context also roots of P determine finite-D extensions since e^p is ordinary p-adic number.
2. Somehow this data should give rise to possibly unique continuous surface. At the level of $H = M^4 \times CP_2$ this is impossible unless the dynamics satisfies besides the action principle

also a huge number of additional conditions reducing the initial value data and/or boundary data to a condition that the surface contains a discrete set of algebraic points.

This condition is horribly strong, much more stringent than holography and even strong holography (SH) implied by the general coordinate invariance (GCI) in TGD framework. However, preferred extremal property at level of $M^4 \times CP_2$ following basically from GCI in TGD context might be equivalent with the reduction of boundary data to discrete data if $M^8 - H$ correspondence [L6] (see <http://tinyurl.com/yd43o2n2>) is accepted. These data would be analogous to discrete data characterizing computer program so that an analog of computationalism would emerge [L3] (see <http://tinyurl.com/y75246rk>).

One can argue that somehow the action of discrete Galois group must have a lift to a continuous flow.

1. The linear superposition of the extension in the field of rationals does not extend uniquely to a linear superposition in the field reals since the expression of real number as sum of units of extension with real coefficients is highly non-unique. Therefore the naïve extension of the extension of Galois group to all points of space-time surface fails.
2. The old idea already due to Riemann is that Galois group is represented as the first homotopy group of the space. The space with homotopy group π_1 has coverings for which points remain invariant under subgroup H of the homotopy group. For the universal covering the number of sheets equals to the order of π_1 . For the other coverings there is subgroup $H \subset \pi_1$ leaving the points invariant. For instance, for homotopy group $\pi_1(S^1) = Z$ the subgroup is nZ and one has $Z/nZ = Z_p$ as the group of n -sheeted covering. For physical reasons it seems reasonable to restrict to finite-D Galois extensions and thus to finite homotopy groups.

$\pi_1 - G$ correspondence would allow to lift the action of Galois group to a flow determined only up to homotopy so that this condition is far from being sufficient.

3. A stronger condition would be that π_1 and therefore also G can be realized as a discrete subgroup of the isometry group of $H = M^4 \times CP_2$ or of M^8 ($M^8 - H$ correspondence) and can be lifted to continuous flow. Also this condition looks too weak to realize the required miracle. This lift is however strongly suggested by Langlands correspondence [K16, K17] (see <http://tinyurl.com/y9x5vkeo>).

The physically natural condition is that the preferred extremal property fixes the surface or at least space-time surface from a very small amount of data. The discrete set of algebraic points in given extension should serve as an analog of boundary data or initial value data.

1. $M^8 - H$ correspondence [L6] (see <http://tinyurl.com/yd43o2n2>) could indeed realize this idea. At the level of M^8 space-time surfaces would be algebraic varieties whereas at the level of H they would be preferred extremals of an action principle which is sum of Kähler action and minimal surface term.

They would thus satisfy partial differential equations implied by the variational principle and infinite number of gauge conditions stating that classical Noether charges vanish for a subgroup of symplectic group of $\delta M_{\pm}^4 \times CP_2$. For twistor lift the condition that the induced twistor structure for the 6-D surface represented as a surface in the 12-D Cartesian product of twistor spaces of M^4 and CP_2 reduces to twistor space of the space-time surface and is thus S^2 bundle over 4-D space-time surface.

The direct map $M^8 \rightarrow H$ is possible in the associative space-time regions of $X^4 \subset M^8$ with quaternionic tangent or normal space. These regions correspond to external particles arriving into causal diamond (CD). As surfaces in H they are minimal surfaces and also extremals of Kähler action and do not depend at all on coupling parameters (universality of quantum criticality realized as associativity). In non-associative regions identified as interaction regions inside CDs the dynamics depends on coupling parameters and the direct map $M^8 \rightarrow CP_2$ is not possible but preferred extremal property would fix the image in the interior of CD from the boundary data at the boundaries of CD.

2. At the level of M^8 the situation is very simple since space-time surfaces would correspond to zero loci for $RE(P)$ or $IM(P)$ (RE and IM are defined in quaternionic sense) of an octonionic polynomial P obtained from a real polynomial with coefficients having values in the field of rationals or in an extension of rationals. The extension of rationals would correspond to the extension defined by the roots of the polynomial P .

If the coefficients are not rational but belong to an extension of rationals with Galois group G_0 , the Galois group of the extension defined by the polynomial has G_0 as normal subgroup and one can argue that the relative Galois group $G_{rel} = G/G_0$ takes the role of Galois group.

It seems that $M^8 - H$ correspondence could allow to realize the lift of discrete data to obtain continuous space-time surfaces. The data fixing the real polynomial P and therefore also its octonionic variant are indeed discrete and correspond essentially to the roots of P .

3. One of the elegant features of this picture is that the at the level of M^8 there are highly unique linear coordinates of M^8 consistent with the octonionic structure so that the notion of a M^8 point belonging to extension of rationals does not lead to conflict with GCI. Linear coordinate changes of M^8 coordinates not respecting the property of being a number in extension of rationals would define moduli space so that GCI would be achieved.

Does this option imply the lift of G to π_1 or to even a discrete subgroup of isometries is not clear. Galois group should have a representation as a discrete subgroup of isometry group in order to realize the latter condition and Langlands correspondence supports this as already noticed. Note that only a rather restricted set of Galois groups can be lifted to subgroups of $SU(2)$ appearing in McKay correspondence and hierarchy of inclusions of hyper-finite factors of type II_1 labelled by these subgroups forming so called ADE hierarchy in 1-1 correspondence with ADE type Lie groups [K23, K14] (see <http://tinyurl.com/ybavqvvr>). One must notice that there are additional complexities due to the possibility of quaternionic structure which bring in the Galois group $SO(3)$ of quaternions.

Remark: After writing this article a considerable progress in understanding of $h_{eff}/h = n$ as number of sheets of Galois covering emerged. By M^8 -duality space-time surface can be seen as zero locus for real or imaginary part (regarding octonions as sums of quaternionic real and imaginary parts) allows a nice understanding of space-time surface as an $h_{eff}/h = n$ -fold Galois covering. M^8 is complexified by adding an imaginary unit i commuting with octonionic imaginary units. Also space-time surface is complexified to 8-D surface in complexified M^8 . One can say that ordinary space-time surface is the “real part” of this complexified space-time surface just like x is the real part of a complex number $x + iy$. Space-time surface can be also seen as a root of n :th order polynomial with n complex branches and the projections of complex roots to “real part” of M^8 define space-time surface as an n -fold covering space in which Galois group acts.

2.8 Connection with infinite primes

The idea about space-time surfaces as zero loci of polynomials emerged for the first time as I tried to understand the physical interpretation of infinite primes [K19], which were motivated by TGD inspired theory of consciousness. Infinite primes form an infinite hierarchy. At the lowest level the basic entity is the product $X = \prod_p p$ of all finite primes. The physical interpretation could be as an analog of fermionic sea with fermion states labelled by finite primes p .

1. The simplest infinite primes are of form $P = X \pm 1$ as is easy to see. One can construct more complex infinite primes as infinite integers of form $nX/r + mr$. Here r is square free integer, n is integer having no common factors with r , and m can have only factors possessed also by r .

The interpretation is that r defines fermionic state obtained by kicking from Dirac sea the fermions labelled by the prime factors of r . The integers n and m define bosonic excitations in which k :th power of p corresponds to k bosons in state labelled by p . One can also construct more complex infinite primes as polynomials of X and having no rational factors. In fact, X becomes coordinate variable in the correspondence with polynomials.

2. This process can be repeated at the next level. Now one introduces product $Y = \prod_P P$ of all primes at the previous level and repeats the same construction. These infinite correspond to polynomials of Y with coefficients given by rational functions of X . Primality means irreducibility in the field of rational functions so that solving Y in terms of X would give algebraic function.
3. At the lowest level are ordinary primes. At the next level the infinite primes are indeed infinite in real sense but have p-adic norms equal to unity. They can be mapped to polynomials $P(x_1)$ with rational coefficients and the simplest polynomials are monomials with rational root. Higher polynomials are irreducible polynomials with algebraic roots. At the third level of hierarchy one has polynomials $P(x_2|x_1)$ of two variables. They are polynomials of x_1 with coefficients with are rational functions of x_1 . This hierarchy can be continued.

One can define also infinite integers as products of infinite primes at various levels of hierarchy and even infinite rationals.

4. This hierarchy can be interpreted in terms of a repeated quantization of an arithmetic supersymmetric quantum field theory with elementary particles labelled by primes at given level of hierarchy. Physical picture suggests that the hierarchy of second quantizations is realized also in Nature and corresponds to the hierarchy of space-time sheets.
5. One could consider a mapping $P(x_n|x_{n-1}|\dots|x_1)$ by a diagonal projection $x_i = x$ to polynomials of single variable x . One could replace x with complexified octonionic coordinate o_c . Could this correspondence give rise to octonionic polynomials and could the connection with second quantization give classical space-time correlates of real quantum states assignable to infinite primes and integers? Even quantum states defining counterparts of infinite rationals could be considered. One could require that the real norm of these infinite rationals equals to one. They would define infinite number of real units with arbitrarily complex number theoretical anatomy. The extension of real numbers by these units would mean huge extension of the notion of real number and one could say that each real point corresponds to platonic defined by these units closed under multiplication.

In ZEO zero energy states formed by pairs of positive and negative energy could correspond to these states physically. The condition that the ratio is unit would have also a physical interpretation in terms of particle content.

6. As already noticed, the notions of analyticity, quaternionicity, and octonionicity could be seen as a manifestation of polynomials in algebras defined by adding repeatedly a new non-commuting imaginary unit to already existing algebra. The dimension of the algebra is doubled in each step so that dimension comes as a power of 2. The algebra of polynomials with real coefficients is commutative and associative. This encourages the crazy idea that the spaces are indeed realized and the generalization of $M^8 - H$ duality holds true at each level. At level k the counterpart for CP_2 (for $k = 3$) would be as moduli space for sub-spaces of dimension 2^{k-1} for which tangent space reduces to the algebra at level $k - 1$. For $k = 2$ CP_1 is the moduli space and could correspond to twistor sphere. Essentially Grassmannian $Gl(2^k, 2^{k-1})$ would be in question. This brings in mind twistor Grassmann approach involving hierarchy of Grassmannians too, which however allows all dimensions. What is interesting that the spinor bundle for space of even dimension d has fiber with dimension $2^{d/2}$.

The number of arguments for the hierarchy of polynomials assignable to the hierarchy of infinite primes increases by one at each step. Hence these two hierarchies are different.

The vanishing of the octonionic polynomials indeed allow a decomposition to products of prime polynomials with roots which in general are algebraic numbers and an exciting possibility is that the prime polynomials have interpretation as counterparts of elementary particles in very general sense.

Infinite primes can be mapped to polynomials and the most natural counterpart for the infinite rational would be as a complexified octonionic rational function $P_1(t)/P_2(t - T)$, where T is real octonion, with coefficients in extension of rationals. This would naturally give the geometry CD. The assignment of opposite boundaries of CD to $P_1(t)$ and $P_2(t - T)$ is suggestive and identification

of zero loci of $IM(P_1)$ and $IM(P_2)$ as incoming and outgoing particles would be natural. The zero and ∞ loci for $RE(P_1/P_2)$ would define interaction between these space-time varieties and should give rise to wormhole contacts connecting them. Note that the linearity of $IM(o_1o_2)$ in $IM(o_i)$ and non-linearity of $RE(o_1o_2)$ in $RE(o_i)$ would be a key element behind this identification. This idea will be discussed in more detail in the section “Gromov-Witten invariants, Riemann-Roch theorem, and Atiyah-Singer index theorem from TGD point of view”.

3 Super variant of octonionic algebraic geometry and space-time surfaces as correlates for fermionic states

Could the octonionic level provide an elegant description of fermions in terms of super variant of octonionic algebraic geometry? Could one even construct scattering amplitudes at the level of M^8 using the variant of the twistor approach discussed in [K15, K18]?

The idea about super-geometry is of course very different from the idea that fermionic statistics is realized in terms of the spinor structure of “world of classical worlds” (WCW) but M^8-H duality could however map these ideas and also number theoretic and geometric vision to each other. The angel of geometry and the devil of algebra could be dual to each other.

In the following I start from the notion of emergence generalized to the vision that entire physics emerges from the notion of number. This naturally leads to an identification of super-variants of various number fields, in particular of complexified octonions. After that super variants of $RE(P) = 0$ and $IM(P) = 0$ conditions are discussed, and the surprising finding is that the conditions might allow only single fermion states localized at strings. This would allow only single particle in the super-multiplet and would mean breaking of SUSY. This picture would be consistent with the earlier H picture about construction of scattering amplitudes [K15, K18]. Finally the problems related to the detailed physical interpretation are discussed.

3.1 About emergence

The notion of emergence is fashionable in the recent day physics, in particular, he belief is that 3-space emerges in some manner. In the sequel I consider briefly the standard view about emergence idea from TGD point of view, then suggest that the emergence in the deepest sense requires emergence of physics from the notion of number and that complexified octonions [L6] [L7, L8, L2, L5] are the most plausible candidate in this respect. After that I will show that number theory generalizes to super-number theory: super-number fields make sense and one can define the notion of super-prime. Every new step of progress creates worry about consistency with the earlier work, now the work done during last months with physics as octonionic algebraic geometry and also this aspect is discussed.

1. The notion of holography is behind the emergence of 3-space and implies that the notion of 2-space is taken as input. This could be justified by conformal invariance.
2. The key idea is that 3-space emerges somehow from entanglement. There is something that must entangle and this something must be labelled by points of space: one must introduce a discretised space. Then one must do some handwaving to make it 3-D - perhaps by arguing that holography based on 2-D holograms is unique by conformal invariance. The next hand-wave would replace this as a 3-D continuous space at infrared limit.
3. How to get space-time and how to get general coordinate invariance? How to get the symmetries of standard model and special relativity? Somehow all this must be smuggled into the theory when the audience is cheated to direct its attention elsewhere. This Münchhausen trick requires a professional magician!
4. One attempt could take as starting point what I call strong form of holography (SH) in which 2-D data determine 4-D physics. Just like 2-D real analytic function determines analytic function of two complex variables in spacetime of 2 complex dimensions by analytic continuation (this hints strongly to quaternions). This is possible if conformal invariance is

generalized to that for light-like 3-surfaces such as light-cone boundary. But the emergence magician should do the same without these.

In TGD one could make this even simpler. Octonionic polynomials and rational functions are obtained from real polynomials of real variable by octonion-analytic continuation. And since polynomials and rational functions P_1/P_2 are in question their values at finite number of discrete points determined them if the orders of P_1 and P_2 are known!

If one accepts adelic hierarchy based on extensions of rationals the coefficients of polynomials are in extensions of rationals and the situation simplifies further. The criticality conditions guaranteeing associativity for external particles is one more simplification: everything becomes discrete. The physics at fundamental level could be incredibly simple: discrete number of points determines space-time surfaces as zero loci for $RE(P)$ or $IM(P)$ (octonions are decomposed to two quaternions gives $RE(o)$ and $IM(o)$).

How this is mapped to physics leading to standard model emerging from the formulation in $M \times CP_2$ This map exists - I call it $M^8 - H$ duality - and takes space-time varieties in Minkowskian sector of complexified octonions to a space-time surface in $M^4 \times CP_2$ coding for standard model quantum numbers and classical fields.

How to get all this without bringing in octonionic embedding space: this is the challenge for the emergence-magician! I am afraid this this trick is impossible. I will however propose a deeper for what emergence is. It would not be emergence of space-time and all physics from entanglement but from the notion of number, which is at the base of all mathematics. This view led to a discovery of the notion of super-number field, a completely new mathematical concept, which should show how deep the idea is.

3.2 Does physics emerge from the notion of number field?

Concerning emergence one can start from a totally different point of view. Even if one gets rid of space as something fundamental from Hilbert space and entanglement, one has not reached the most fundamental level. Structures like Hilbert space, manifold, etc. are not fundamental mathematical structures: they require the notion of number field. Number field is the fundamental notion.

Could entire physics emerge from the notion of number field alone: space-time, fermions, standard model interactions, gravitation? There are good hopes about this in TGD framework if one accepts $M^8 - H$ duality and physics as octonionic algebraic geometry! One could however argue that fermions do not follow from the notion of number field alone. The real surprise was that formalizing this more precisely led to a realization that the very notion of number field generalizes to what one could call super-number field!

3.2.1 Emergence of physics from complexified octonionic algebraic geometry

Consider first the situation for number fields postponing the addition of attribute “super” later.

1. Number field endowed with basic arithmetic operations $+$, $-$, \cdot , $/$ is the basic notion for anyone wanting to make theoretical physics. There is a rich repertoire of number fields. Finite fields, rationals and their extensions, real numbers, complex numbers, quaternions, and octonions. There also p-adic numbers and their extensions induced by extensions of rationals and fusing into adèle forming basic structure of adelic physics. Even the complex, quaternionic, and octonionic rationals and their extensions make sense. p-Adic variants of say octonions must be however restricted to have coefficients belonging to an extension of rationals unless one is willing to give up field property (the p-adic analog of norm squared can vanish in higher p-adic dimensions so that inverse need not exist).

There are also function fields consisting of functions with local arithmetic operations. Analytic functions of complex variable provides the basic example. If function vanishes at some point its inverse element diverges at the same point. Function fields are derived objects rather than fundamental.

2. Octonions are the largest classical number field and are therefore the natural choice if one wants to reduce physics to the notion of number. Since one wants also algebraic extensions of rationals, it is natural to introduce the notion of complexified octonion by introducing an additional imaginary unit - call it i , commuting with the 7 octonionic imaginary units I_k . One obtains complexified octonions.

That this is not a global number field anymore turns out to be a blessing physically. Complexified octonion $z_k E^k$ has $z_k = z_k + iy_k$. The complex valued norm of octonion is given by $z_0^2 + \dots z_7^2$ (there is no conjugation involved). The norm vanishes at the complex surface $z_0^2 + \dots z_7^2 = 0$ defining a 7-D surface in 7-D O_c (the dimension is defined in complex sense). At this surface - complexified light-cone boundary - number field theory property fails but is preserved elsewhere since one can construct the inverse of octonion.

At the real section M^8 (8-D Minkowski space with one real (imaginary) coordinate and 7 imaginary (real) coordinates the vanishing takes place also. This surface corresponds to the 7-D light-cone boundary of 8-D Minkowskian light-cone. This suggests that light-like propagation is basically due to the complexification of octonions implying local failure of the number field property. Same happens also in other real sections with $0 < n < 8$ real coordinates and $0 < m = 8 - n < 8$ imaginary coordinates and one obtains variant of light-cone with different signatures. Euclidian signature corresponding to $m = 0$ or $m = 8$ is an exception: light-cone boundary reduces to single point in this case and one has genuine number field - no propagation is possible in Euclidian signature.

Similar argument applies in the case of complexified quaternions Q_c and complexified complex numbers $z_1 + z_2 I \in C_c$, where I is octonionic imaginary unit. For Q_c one obtains ordinary 3-D light-cone boundary in real section and 1-D light-cone boundary in the case of C_c . It seems that physics demands complexification! The restriction to real sector follows from the requirement that norm squared reduces to a real number. All real sectors are possible and I have already considered the question whether this should be taken as a prediction of TGD and whether it is testable.

3.2.2 Super-octonionic algebraic geometry

There is also a natural generalization of octonionic TGD to super-octonionic TGD based on octonionic triality. $SO(1,7)$ allows besides 8-D vector representations also spinor representations 8_c and $\bar{8}_c$. This suggests that super variant of number field of octonions might make sense. One would have $o = o_8 + o_{c,8} + \bar{o}_{c,8}$.

1. Should one combine o_8 , $o_{c,8}$ and $\bar{o}_{c,8}$ to a coordinate triplet $(o_8, o_{c,8}, \bar{o}_{c,8})$ as done in supersymmetric theories to construct super-fields? The introduction of super-fields as primary dynamical variables is a good idea now since the very idea is to reduce physics to algebraic geometry at the level of M^8 . Polynomials of super-octonions defining space-time varieties as zero loci for their real or imaginary part in quaternionic sense could however take the role of super fields. Space-time surface would correspond to zero loci for $RE(P)$ or $IM(P)$.
2. The idea about super-octonions should be consistent with the idea that we live in a complexified number field. How to define the notion of super-octonion? The tensor product $8 \otimes 8_c$ contains 8_c and $8 \otimes \bar{8}_c$ contains $8_{\bar{c}}$ and one can use Glebsch-Gordan coefficients to contract o and θ_c and o and $\theta_{\bar{c},n}$. The tensor product of 8_c and $8_{\bar{c}}$ defined using structure constants defining octonion product gives 8. Therefore one must have

$$o_s = o + \Psi_c \times \theta_{\bar{c}} + \Psi_{\bar{c}} \times \theta_c , \quad (3.1)$$

where the products are octonion products. Super parts of super-coordinates would not be just Grassmann numbers but octonionic products of Grassmann numbers with octonionic spinors in 8_c and $\bar{8}_c$. This would bring in the octonionic analogs of spinor fields into the octonionic geometry.

This seems to be consistent with super field theories since octonionic polynomials and even rational functions would give the analogs of super-fields. What TGD would provide would be an algebraic geometrization of super-fields.

3. What is the meaning of the conditions $RE(P) = 0$ and $IM(P) = 0$ for super-octonions? Does this condition hold true for all $d_G = 2^{16}$ super components of $P(o_s)$ or is it enough to pose the condition only for the octonionic part of $P(o)$? In the latter case Ψ_c and $\Psi_{\bar{c}}$ would be free and this does not seem sensible and does not conform with octonionic super-symmetry. Therefore the first option will be studied in the sequel.

If super-octonions for a super variant of number field so that also inverse of super-octonion is well-defined, then even rational functions of complexified super-octonions makes sense and poles have interpretation in terms of 8-D light-fronts (partonic orbits at level of H). The notion must make sense also for other classical number fields, finite fields, rationals and their extensions, and p -adic numbers and their extensions. Does this structure form a generalization of number field to a super counter part of number field? The easiest manner to kill the idea is to check what happens in the case of reals.

1. The super-real would be of form $s = x + y\theta$, $\theta^2 = 0$. Sum and product are obviously well-defined. The inverse is also well-defined and given by $1/s = (x - y\theta)/x^2$. Note that for complex number $x + iy$ the inverse would be $\bar{z}/z\bar{z} = (x - yi)/(x^2 + y^2)$. The formula for super-inverse follows from the same formula as the inverse of complex number by defining conjugate of super-real s as $\bar{s} = x - y\theta$ and the norm squared of s as $|s|^2 = s\bar{s} = x^2$.

One can identify super-integers as $N = m + n\theta$. One can also identify super-real units as number of unit norm. Any number $1_n = 1 + n\theta$ has unit norm and the norms form an Abelian group under multiplication: $1_m 1_n = 1_{m+n}$. Similar non-uniqueness of units occurs also for algebraic extensions of rationals.

2. Could one have super variant of number theory? Can one identify super-primes? Super-norm satisfies the usual defining property $|xy| = |x||y|$. Super-prime is defined only apart from the multiplicative factor 1_m giving not contribution to the norm. This is not a problem but a more rigorous formulation leads to the replacement of primes with prime ideals labelled by primes already in the ordinary number theory.

If the norm of super-prime is ordinary prime it cannot decompose to a product of super-primes. Not all super-primes having given ordinary prime as norm are however independent. If super-primes $p + n\theta$ and $p + m\theta$ differ by a multiplication with unit $1_r = 1 + r\theta$, one has $n - m = pr$. Hence there are only p super-primes with norm p and they can be taken $p_s = p + k\theta$, $k \in \{0, p - 1\}$. A structure analogous to a cyclic group Z_p emerges.

Note that also θ is somewhat analogous to prime although its norm is vanishing.

3. Just for fun, one can ask what is the super counterpart of Riemann Zeta. Riemann zeta can be regarded as an analog of thermodynamical partition function reducing to a product for partition functions for bosonic systems labelled by primes p . The contribution from prime p is factor $1/(1 - p^{-s})$. p^{-s} is analogous to Boltzmann weight $N(E)\exp(-E/T)$, where $N(E)$ is number of states with energy E . The degeneracy of states labelled by prime p is for ordinary primes $N(p) = 1$. For super-primes the degeneracy is $N(p) = p$ and the weight becomes $1/(1 - N(p)p^{-s}) = 1/(1 - p^{-s+1})$. Super Riemann zeta is therefore $\zeta(s - 1)$ having critical line at $s = 3/2$ rather than at $s = 1/2$ and trivial zeros at real points $s = -1, -3, -5$, rather than at $s = -2, -4, -6, \dots$

There are good reasons to expect that the above arguments work also for algebraic extensions of super-rationals and in fact for all number fields, even for super-variants of complex numbers, quaternions and octonions. This because the conditions for invertibility reduce to that for real numbers. One would have a generalization of number theory to super-number theory! Net search gives no references to anything like this. Perhaps the generalization has not been noticed because the physical motivation has been lacking. $M^8 - H$ duality would imply that entire physics, including fermion statistics, standard model interactions and gravitation reduces to the notion of number in accordance with number theoretical view about emergence.

3.2.3 Is it possible to satisfy super-variants of $IM(P) = 0$ and $RE(P) = 0$ conditions?

Instead of super-fields one would have a super variant of octonionic algebraic geometry.

1. Super variants of the polynomials and even rational functions make sense and reduce to a sum of octonionic polynomials $P_{kl}\theta_1^k\theta_2^l$, where the integers k and l would be tentatively identified as fermion numbers and θ_k is a shorthand for a monomial of k different thetas. The coefficients in $P_{kl} = P_{kl,n}o^n$ would be given by $P_{kl,n} = P_{n+k+l}B(n+k+l, k+l)$, where $B(r, s) = r!/(r-s)!s!$ is binomial coefficient. The space-time surfaces associated with P_{kl} would be different and they need not be simultaneously critical, which could give rise to a breaking of supersymmetry.

One would clearly have an upper bound for k and l for given CD. Therefore these many-fermion states must correspond to fundamental particles rather than many-fermion Fock states. One would obtain bosons with non-vanishing fermion numbers if the proposed identification is correct. Octonionic algebraic geometry for single CD would describe only fundamental particles or states with bounded fermion numbers. Fundamental particles would be indeed fundamental also geometrically.

2. One can also now define space-time varieties as zero loci via the conditions $RE(P_s)(o_s) = 0$ or $IM(P_s)(o_s) = 0$. One obtains a collection of 4-surfaces as zero loci of P_{kl} . One would have a correlation with between fermion content and algebraic geometry of the space-time surface unlike in the ordinary super-space approach, where the notion of the geometry remains rather formal and there is no natural coupling between fermionic content and classical geometry. At the level of H this comes from quantum classical correspondence (QCC) stating that the classical Noether charges are equal to eigenvalues of fermionic Noether charges.

In the definition of the first variant of super-octonions I followed the standard idea about what super-coordinates assuming that the super-part of super-octonion is just an anti-commuting Grassmann number without any structure: I just replaced o with $o + \theta_k E^k + \bar{\theta}_k E^k$ regarding θ_k as anticommuting coordinates. Now θ_k receives octonionic coefficient: $\theta_k \rightarrow o_k \theta_k$. θ_k is now analogous to unit vector.

For the super-number field inspired formulation the situation is different since one assigns independent octonionic coordinates to anticommuting degrees of freedom. One has linear space with partially anti-commutative basis. O_c is effectively replaced with O_c^3 so that one has $8+8+8=24$ -dimensional Cartesian product (it is amusing that the magic dimension 24 for physical polarizations of bosonic string models emerges).

What is the number of equations in the new picture? For N super-coordinates one has 2^N separate monomials analogous to many-fermion states. Now one has $N = 8 + 8 = 16$ and this gives 2^{16} monomials! In the general case $RE = 0$ or $IM = 0$ gives 4 equations for each of the $d_G = 2^{16}$ monomials: the number of equations $RE = 0$ or $IM = 0$ is 4×2^{16} and exceeds the number $d_O = 24$ of octonion valued coordinates. In the original interpretation these equations were regarded as independent and gave different space-time variety for each many-fermion state.

In the new framework these equations cannot be treated independently. One has 24 octonionic coordinates and 2^{16} equations. In the generic case there are no solutions. This is actually what one hopes since otherwise one would have a state involving superposition of many-fermion states with several fermion numbers.

The freedom to pose constraints on the coefficients of Grassmann parameters however allows to reduce degrees of freedom. All coefficients must be however expressible as products of $3 \times 8 = 24$ components of super-octonion.

1. One can have solutions for which both 8_c part and $\bar{8}_c$ parts vanish. This gives the familiar 4 equations for 8 variables and 4-surfaces.
2. Consider first options, which fail. If 8_c - or $\bar{8}_c$ part vanishes one has $d_G = 2^8$ and $4 \times d_G = 4 \times 64$ equations for $d_O = 8 + 8 = 16$ variables having no solutions in the generic case. The restriction of 8_c to its 4-D quaternionic sub-space would give $d_O = 4$ and $4d_G = 4 \times 2^4 = 64$ conditions and 16 variables. The reduction to complex sub-space $z_1 + z_2 I$ of super-octonions would give $d_O = 2^2$ and $4 \times 2^2 = 16$ conditions for $8 + 2 = 10$ variables.
3. The restriction to 1-D sub-space of super-octonions would give $4 \times 2^1 = 8$ conditions and $8 + 1 = 9$ variables. Could the solution be interpreted as 1-D fermionic string assignable to

the space-like boundary of space-time surface at the boundary of CD? Skeptic inside me asks whether this could mean the analog of $\mathcal{N} = 1$ SUSY, which is not consistent with H picture.

Second possibility is restriction to light-like subspace for which powers of light-like octonion reduce effectively to powers of real coordinate. Fermions would be along light-lines in M^8 and along light-like curves in H . The powers of super-octonion have super-part, which belongs to the 1-D super-space in question: only single fermion state is present besides scalar state.

4. There are probably other solutions to the conditions but the presence of fermions certainly forces a localization of fermionic states to lower-dimensional varieties. This is what happens also in H picture. During years the localization of fermion to string worlds sheets and their boundaries has popped up again and again from various arguments. Could one hope that super-number theory provides the eventual argument.

But how could one understand string world sheets in this framework? If they do not carry fermions at H-level, do they appear naturally as 2-D structures in the ordinary sense?

To sum up, although many details must be checked and up-dated, super-number theory provides and extremely attractive approach promising ultimate emergence as a reduction of physics to the notion of number. When physical theory leads to a discovery of new mathematics, one must take it seriously.

3.3 About physical interpretation

Super-octonionic algebraic geometry should be consistent with the H picture in which baryon and lepton numbers as well as other standard model quantum numbers can be understood. There are still many details, which are not properly understood.

3.3.1 The interpretation of theta parameters

The interpretation of theta parameters is not completely straightforward.

1. The first interpretation is that θ_c and $\theta_{\bar{c}}$ correspond to objects with opposite fermion numbers. If this is not the case, one could perhaps define the conjugate of super-coordinate as octonionic conjugate $\bar{o}_s = \bar{o} + \bar{\theta}_1 + \bar{\theta}_2$. This looks ugly but cannot be excluded.

There is also the question about spinor property. Octonionic spinors are 2-spinors with octonion valued components. Could one say that the coefficients of octonion units have been replaced with Grassmann numbers and the entire 2-component spinor is represented as a pair of θ_c and $\theta_{\bar{c}}$? The two components of spinor in massless theories indeed correspond to massless particle and its antiparticle.

2. One should obtain particles and antiparticles naturally as also separately conserved baryon and lepton numbers (I have also considered the identification of hadrons in terms of anyonic bound states of leptons with fractional charges).

Quarks and leptons have different coupling to the induced Kähler form at the level of H . It seems impossible to understand this at the level of M^8 , where the dynamics is purely algebraic and contains no gauge couplings.

The difference between quarks and leptons is that they allow color partial waves with triality $t = \pm 1$ and triality $t = 0$. Color partial waves correspond to wave functions in the moduli space CP_2 for $M_0^4 \supset M_0^2$. Could the distinction between quarks and leptons emerge at the level of this moduli space rather than at the fundamental octonionic level? There would be no need for gauge couplings to distinguish between quarks and leptons at the level of M^8 . All couplings would follow from the criticality conditions guaranteeing 4-D associativity for external particles (on mass shell states would be critical).

If so, one would have only the super octonions and θ_c and $\theta_{\bar{c}}$ would correspond to fermions and antifermions with no differentiation to quarks or leptons. Fermion number conservation would be coded by the Grassmann algebra. Quantum classical correspondence (QCC) however suggests that it should be possible to distinguish between quarks and leptons already at M^8 level. Is it really enough that the distinction comes at the level of moduli space for CDs?

One can imagine also other options but they have their problems. Therefore this option will be considered in the sequel.

3.3.2 Questions about quantum numbers

The first questions relate to fermionic statistics.

1. Do super-octonions really realize fermionic statistics and how? The polynomials of super-octonions can have only finite degree in θ and θ_c . One can say that only finite number of fermions are possible at given space-time point. As found, the conditions $IM(P) = 0$ and $RE(P) = 0$ might allow only single fermion strings as solutions perhaps assignable to partonic 2-surfaces.

Can one allow for given CD arbitrary number of this kind of points as the idea that identical fermions can reside at different points suggests? Or is the number of fermions finite for given CD or correspond to the highest degree monomial of θ and θ_c in P ?

Finite fermion number of CD looks somewhat disappointing at first. The states with high fermion numbers would be described in terms of Cartesian products just like in condensed matter physics. Note however that space-time varieties with different octonionic time axes must be in any case described in this manner. It seems possible to describe the interactions using super-space delta functions stating that the interaction occur only in the intersection points of the space-time surfaces. The delta function would have also super-part as in SUSYs.

2. As found, the theta degree effectively reduces to $d = 1$ for the pointlike solutions, which by above argument are the only possible solutions besides purely bosonic solutions. Only single fermion would be allowed at given point. I have already earlier considered the question whether the partonic 2-surfaces can carry also many-fermion states or not [K15, K18], and adopted the working hypothesis that fermion numbers are not larger than 1 for given worm-hole throat, possibly for purely dynamical reasons. This picture however looks too limited. The many fermion states might not however propagate as ordinary particles (the proposal has been that their propagator pole corresponds to higher power of p^2).

The M^8 description of particle quantum numbers should be consistent with H description.

1. Can octonionic super geometry code for the quantum numbers of the particle states? It seems that super-octonionic polynomials multiplied by octonionic multi-spinors inside single CD can code only for the electroweak quantum numbers of fundamental particles besides their fermion and anti-fermion numbers. What about color?

As already suggested, color corresponds to partial waves in CP_2 serving as moduli space for $M_0^4 \supset M_0^2$. Also four-momentum and angular momentum are naturally assigned with the translational degrees for the tip of CD assignable with the fundamental particle.

2. Quarks and leptons have different trialities at H level. How can one understand this at M^8 level. Could the color triality of fermion be determined by the color representation assignable to the color decomposition of octonion as $8 = 1 + 1 + 3 + \bar{3}$. This decomposition occurs for all 3 terms in the super-octonion. Could the octet in question correspond to the term $D(8 \otimes 8_c; 8_c)_k^{mn} o_{c,m} \theta_{c,n} E^k$ and analogous $\theta_{\bar{c}}$ term in super octonion. Only this kind of term survives from the entire super-octonion polynomial at fermionic string for the solutions found.
3. There is however a problem: $8 = 1 + 1 + 3 + \bar{3}$ decomposition is not consistent with the idea that θ_c and $\theta_{\bar{c}}$ have definite fermion numbers. Quarks appear only as 3, not $\bar{3}$. Why $\bar{3}$ from θ term and 3 from $\theta_{\bar{c}}$ term should drop out as allowed single fermion state?

There are also other questions.

1. What about twistors in this framework? $M^4 \times CP_1$ as twistor space with CP_1 coding for the choice of $M_0^2 \subset M_0^4$ allows projection to the usual twistor space CP_3 . Twistor wave functions describing spin elegantly would correspond to wave functions in the twistor space and one

expects that the notion of super-twistor is well-defined also now. The 6-D twistor space $SU(3)/U(2) \times U(1)$ of CP_2 would code besides the choice of $M_0^4 \supset M_0^2$ also quantization axis for color hypercharge and isospin.

2. The intersection of space-time surfaces with S^6 giving analogs of partonic 2-surfaces might make possible for two sparticle lines to fuse to form a third one at these surfaces. This would define sparticle 3-vertex in very much the same manner as in twistor Grassmann approach to $\mathcal{N} = 4$ SUSY.

H -picture however supports the alternative option that sparticles just scatter but there is no contact interaction defining analog of 3-vertex. If the lines can carry only single fermion, the H picture about twistor diagrams [K15, K18] would be realized also at the level of M^8 ! This means breaking of SUSY since only single fermion states from the octonionic SUSY multiplet are realized. This would provide and easy - perhaps too easy - explanation for the failure to find SUSY at LHC.

3. What about the sphere S^6 serving as the moduli space for the choices of M_+^8 ? Should one have wave functions in S^6 or can one restrict the consideration to single M_+^8 ? As found, one obtains S^6 also as the zero locus of $Im(P) = 0$ for some radii identifiable as values t_n of time coordinates given as roots of $P(t)$: as matter of fact, $S^6(t_n)$ is a solution of both $RE(P) = 0$ and $IM(P) = 0$. Can one identify the intersections $X^4 \cap S^6$ as 2-D as partonic 2-surfaces serving as topological vertices?

4 Could scattering amplitudes be computed in the octonionic framework?

Octonionic algebraic geometry might provide incredibly simple framework for constructing scattering amplitudes since now variational principle is involved and WCW reduces to a discrete set of points in extension of rationals.

4.1 Could scattering amplitudes be computed at the level of M^8 ?

It would be extremely nice if the scattering amplitudes could be computed at the octonionic level by using a generalization of twistor approach in ZEO finding a nice justification at the level of M^8 . Something rather similar to $\mathcal{N} = 4$ twistor Grassmann approach suggests itself.

1. In ZEO picture one would consider the situation in which the passive boundary of CD and members of state pairs at it appearing in zero energy state remain fixed during the sequence of state function reductions inducing stepwise drift of the active boundary of CD and change of states at it by unitary U-matrix at each step following by a localization in the moduli space for the positions of the active boundary.
2. At the active boundary one would obtain quantum superposition of states corresponding to different octonionic geometries for the outgoing particles. Instead of functional integral one would have sum over discrete points of WCW. WCW coordinates would be the coefficients of polynomial P in the extension of rationals. This would give undefined result without additional constraints since rationals are a dense set of reals.

Criticality however serves as a constraint on the coefficients of the polynomials and is expected to realize finite measurement resolution, and hopefully give a well defined finite result in the summation. Criticality for the outgoing states would realize purely number theoretically the cutoff due to finite measurement resolution and would be absolutely essential for the finiteness and well-definedness of the theory.

4.2 Interaction vertices for space-time surfaces with the same CD

Consider interaction vertices for space-time surfaces associated with given CD. At the level of H the fundamental interactions vertices are partonic 2-surfaces at which 3 light-like partonic orbits meet. The incoming light-like sparticle lines scatter at this surface and they are not assumed to

meet at single vertex. This assumption is motivated because it allows to avoid infinities but one must be ready to challenge it. It is essential that wormhole throats appear in pairs assignable to wormhole contacts and also contacts form pairs by the conservation of Kähler magnetic flux.

What could be the counterpart of this picture at level of M^8 ?

1. The simplest interaction could be associated with the common stable intersection points of the space-time regions. By dimensional consideration these intersections are stable and form a discrete set. This would however allow only 2-vertices involved in processes like mixing of states. In the generic case the intersection would consist of discrete points.
2. A stronger condition would be that these points belong to the extension of rationals defining adeles or is extension defined by the polynomial P . This would conform with the idea that scattering amplitudes involve only data associated with the points in the extension. The interaction points could be ramified points at which the action of a subgroup H of Galois group G would leave sheets of the Galois covering invariant so that some number of sheets would touch each other. I have discussed this proposal in [L4]. These points could be seen as analogs of interaction points in QFT description in terms of n -point functions and the sum over polynomials would give rise to the analog over integral over different n -point configurations.
3. A possible interpretation is that if the subgroup $H \subset G$ has k -elements the vertex represents meeting of k sparticle lines and thus k -vertex would be in question. This picture is not what the H view about twistor diagrams [K18] suggests: in these diagrams sparticle lines at the light-like orbits of partonic 2-surfaces do not meet at single point but only scatter at partonic 2-surface, where three light-like orbits of partonic 2-surfaces meet.
4. An alternative interpretation is that k -vertex describes the decay of particle to k fractional particles at partonic 2-surfaces and has nothing do with the usual interaction vertex.

This proposal need not describe usual particle scattering. Could the intersection of space-time varieties defined as zero loci for $RE(P_i)$ and $IM(P_i)$ with the special solutions $S^6(t_n)$ and $CD = M_+^4 \cap M_-^4$ define the loci of interaction? It is difficult to believe that these special solutions could be only a beauty spot of the theory. $X^2 = X^4 \cap S^6(t_n)$ is 2-D and $X^0 = X^4 \cap CD$ consists of discrete points.

Consider now the possible role of the singular ($RE(P) = IM(P) = 0$) maximally critical surface $S^6(t_n)$ in the scattering.

1. As already found, the 6-D spheres S^6 with radii t_n given by the zeros of $P(t)$ are universal and have interpretation as $t = t_n$ snapshots of 7-D spherical light front projection to $t = t_n$ 3-balls as cross sections of 4-D CD. Could the 2-D intersection $X^2 = X^4 \cap S^6(t_n)$ play a fundamental role in the description of interaction vertices?
2. Suppose that 3-vertices realize the dynamical realization of octonionic SUSY predicting large number of sparticles. Could one understand in this framework the 3-vertex for the orbits X_i^3 of partonic 2-surfaces meeting each other along their 2-D end defining partonic 2-surface and understand how 3 fermions lines meet at single point in this picture?
3. Assume that 3 partonic orbits X_i^3 , $i = 1, 2, 3$ meet at $X^2 = X^4 \cap S^6(t_n)$. That this occurs could be part of boundary conditions, which should follow from interaction consistency. If fermions just *go through* the X_i^3 in time direction they cannot meet at single point in the generic case. If the sparticle lines however can *move along* X^2 - maybe due the fact that an intersection $X^2 = X^4 \cap S^6(t_n)$ is in question - they intersect in the generic case and fuse to a third fermion line. Note that this portion of fermion line would be space-like whereas outside X^2 the line would be light-like. This can be used as an objection against the idea.

The picture allowing 3-vertices would be different from H picture in which fermion lines only scatter and only 2+2 fermion vertex assignable to topological 3-vertex is fundamental.

1. One would have 2 wormhole contacts carrying fermion and third one carrying fermion anti-fermion pair at its opposite throats and analogous to boson. Of course, one can reproduce the earlier picture by giving up the condition about supersymmetric fermionic 3-vertex. On the other hand, the idea that interactions occur only at discrete points in extension of rationals is extremely attractive.
2. The surprising outcome from the construction of solutions of super-variants of $RE(P) = 0$ and $IM(P) = 0$ conditions was that if the superpart of super-octonion is non-vanishing, the variety can be only 1-D string like entity carrying one-fermion state. This does allow strings with higher fermion number so that the 3-vertex would not be possible! This suggests that fermionic lines appear as sub-varieties of space-time variety.

If so the original picture [K18] applying at the level of H applies also at the level of M^8 . SUSY is broken dynamically allowing only single fermion states localized at strings and scattering of these occurs by classical interactions at the partonic 2-surfaces defining the topological vertices.

3. The only manner to have a point/line containing sparticle with higher fermion number would be as a singularity along which several branches of super-variety degenerate to single point/line: each variety would carry one fermion line. Unbroken octonionic SUSY would characterize singularities of the space-time varieties, which would be unstable so that SUSY would break. Singularities are indeed critical and thus unstable and also tend to possess enhanced symmetries.

What could be the interpretation of $X^0 = X^4 \cap CD$? For instance, could it be that these points code for 4-momenta classically so that quantum classical correspondence (QCC) would be realized also at the level of M^8 although there are no Noether charges now. But what about angular momenta? Could twistorialization realized in terms of the quaternionic structure of M_0^4 help here. What is the role of the intersections of 6-D twistor bundle of X^4 with 6-D twistor bundle of M_0^4 consisting of discrete points?

The interaction vertex would involve delta function telling that the interacting space-time varieties or their regions touch at same point of M^8 . Delta function in theta parameter degrees of freedom and Grassmann integral over them would be also involved and guarantee fermion number conservation. Vertex factor should be determined by arguments used in Grassmannian twistor approach. I have developed a proposal in [K18] but this proposal allows only fermion number ± 1 at fermion lines. Now all members of the multiplet would be allowed.

4.3 How could the space-time varieties associated with different CDs interact?

The interaction of space-time surfaces inside given CD is well-defined in the octonionic algebraic geometry. The situation is not so clear for different CDs for which the choice of the origin of octonionic coordinates is in general different and polynomial bases for different CDs do not commute nor associate.

The intuitive expectation is that 4-D/8-D CDs can be located everywhere in M^4/M^8 . The polynomials with different origins neither commute nor are associative. Their sum is a polynomial whose coefficients are not real. How could one avoid losing the extremely beautiful associative and commutative algebra of polynomials?

1. Should one assume that the physics observable by single conscious observer corresponds to single CD defining the perceptive field of this observer [L10].
2. Or should one give up associativity and allow products (but not sums since one should give up the assumption that the coefficients of polynomials are real) of polynomials associated with different CDs as an analog for the formation of free many-particle states.

Consider first what happens for the single particle solutions defined as solutions of either $RE(P_i) = 0$ or $IM(P_i) = 0$.

1. The polynomials associated with different 8-D CDs do not commute nor associate. Should one allow their products so that one would still *effectively* have a Cartesian product of commutative and associative algebras? This would realize non-commutative and non-associative physics emerging in conformal field theories also at the level of space-time geometry.
2. If the CDs differ by a *real* (time) translation $o_2 = o_1 + T$ one still obtains $IM(P_1) = 0$ and $IM(P_2) = 0$ as solutions to $IM(P_1P_2) = 0$. This applies also to states with more particles. The identification would be in terms of external particles. For $RE(P_1P_2) = 0$ this is not the case. If the interior of CD corresponds to $RE(P_1P_2) = 0$, the dynamics in the interior is not only non-trivial but also non-commutative and non-associative. Non-trivial interaction would be obtained even without interaction terms in the polynomial vanishing at the boundaries of CD!

Could one consider allowing only CDs with tips at the same real axis but having all sizes scales? This hierarchy of CD would characterize a particular hierarchy of conscious observers - selves having sub-selves (sub-CDs) [L10]. The allowance of only these CD would be analogous to a fixing of quantization axes.

3. What happens if one allows CDs differing by arbitrary octonion translation? Consider external particles. For P_1 and P_2 RE and IM are defined for different decompositions $o_i = RE(o_i) + n_i Im(o_i)$, where $n_i, i = 1, 2$ is a unit octonion.

What decomposition should one use for P_1P_2 ? The decomposition for P_1 or P_2 or some other decomposition? One can express $P_2(o_2)$ using o_1 as coordinate but the coefficients multiplying powers of o_1 from *right* would not be real numbers anymore implying $IM(P_2)_1 \neq IM(P_2)_2$. $IM(P_2)_1 = 0$ makes sense but the presence of particle 1 would have affected particle 2 or vice versa.

Could one argue that the coordinate systems satisfying the condition that some external particles described by P_i have real coefficients and perhaps serving in the role of observers are preferred? Or could one imagine that o_{12} is a kind of center of mass coordinate? In this case the 4-varieties associated with both particles would be affected. What is clear that the choice of the octonionic coordinate origin would affect the space-time varieties of external particles even if they could remain associative/critical.

4. Are there preferred coordinates in which criticality is preserved? For instance, can one achieve criticality for P_2 on coordinates of o_1 if P_1 is critical. Could one see this as a kind of number theoretic observer effect at the level of space-time geometry?

Remark: $P_i(o)$ would reduce to a real polynomial at light-like rays with origin for o_i irrespective of the octonionic coordinate used so that the spheres S_i^6 with origin at the origin of o_i as solutions of $P_i(o) = 0$ would not be lost.

If one does not give up associativity and commutativity for polynomials, how can one describe the interactions between space-time surfaces inside different CDs at the level of M^8 ? The following proposal is the simplest one that one can imagine by assuming that interactions take place at discrete points of space-time surfaces with coordinates belonging an extension of rationals.

1. The most straightforward manner would be to introduce Cartesian powers of O and CD:s inside these powers to describe the interaction between CDs with different origin. This would be analogous to what one does in condensed matter physics. What seems clear is that $M^8 - H$ correspondence should map all the factors of $(M^8)^n$ to the same $M^4 \times CP_2$ by a kind of diagonal projection.

In topological 3-parton vertex X^2 three light-like partonic orbits along X^4 would meet. X^2 would be the contact of X^4 with S^6 associated with second 8-D CD. Together with SH this gives hopes about an elegant description of interactions in terms of connected space-time varieties.

2. The intersection $X_i^4 \cap X_j^4$ consists of discrete set of points. This would suggest that the interaction means transfer of fermion between X_1^4 and X_2^4 . The intersection of $X = S_1^6(t_m) \cap S_2^6(t_n)$ is 4-D and space-like. The intersection $X_i^4 \cap X$ consists of discrete points could these discrete points allow to construct interaction vertices.

To make this more concrete, assume that the external particles outside the interaction CD (CD_{int}) defining the interaction region correspond to associative (or co-associative) space-time varieties with different CDs.

Remark: CDs are now 8-dimensional.

1. One can assign the external particles to the Cartesian factors of $(M^8)^n$ giving (P_1, \dots, P_n) just like one does in condensed matter physics for particles in 3-space E^3 . Inside CD_{int} the Cartesian factors would fuse to single factor and instead of Cartesian product one would have the octonionic product $P = \prod P_i$ plus the condition $RE(P) = 0$ (or $IM(P) = 0$: one should avoid too strong assumptions at this stage) would give to the space-time surface defining the interaction region.
2. $RE(P) = 0$ and $IM(P) = 0$ conditions make sense even, when the polynomials do not have origin at common real axis and give rise to 4 conditions for 8 polynomials of 8 complexified octonion components P^i . It is not possible to reduce the situation at the light-like boundaries of 8-D light-cone to a vanishing of polynomial $P(t)$ of real coordinate t anymore, and one loses the the surfaces S_i^6 as special solutions and therefore also the partonic 2-surfaces $X_i^2 = X^4 \cap S_i^6$. Should one assign all X_i^2 with the intersections of external particles with the two boundaries δ_{\pm} CD of CD defining the interaction region. They would intersect δ_{\pm} CD at highly unique discrete points defining the sparticle interaction vertices. By 7-dimensionality of δ_{\pm} CD the intersection points would be at the boundaries of 4-D CD and presumably at light-like partonic orbits at which the induced metric is singular at H side at least just as required by H picture.

The most general external single-sparticle state would be defined by a product P of mutually commuting and associating polynomials with tips of CD along common real axis and satisfying $IM(P_i) = 0$ or $RE(P_i) = 0$. This could give both free and bound states of constituents.

3. Different orders and associations for $P = \prod P_i$ give rise to different interaction regions. This requires a sum over the scattering amplitudes $\sum_p T(\prod_i P_{p(i)})$ associated with the permutations $p: (1, \dots, n) \rightarrow (p(1), \dots, p(n))$ and $T = \sum_p U(p)T(P_{p(1)} \dots P_{p(n)})$ ($T(AB) + T(BA)$) in the simplest case with suitable phase factors $U(p)$. Note that one does *not* have a sum over the polynomials $P_{p(1)} \dots P_{p(n)}$ but over the scattering amplitudes associated with them.
4. Depending on the monomial of theta parameters in super-octonion part of P_i , one has plus or minus signs under the exchange of P_i and P_j . One can also have braid group as a lift of the permutation group. In this case given contribution to the scattering amplitude has a phase factor depending on the permutation (say $T = T(AB) + \exp(i\theta)T(BA)$).

One must also form the sum $T = \sum_{Ass} U(Ass)T(Ass(P))$ over all associations for a given permutation with phase factors $U(Ass)$. Here $T = T((AB)C) + UT(A(BC))$, U phase factor, is the simplest case. One has “association statistics” as the analog of braid statistics. Permutations and associations have now a concrete geometric meaning at the level of space-time geometry - also at the level of H .

5. The geometric realization of permutations and associations could relate to the basic problem encountered in the twistorial construction of the scattering amplitudes. One has essentially sum over the cyclic permutations of the external particles but does not know how to construct the amplitudes for general permutations, which correspond to non-planar Feynman diagrams. The geometric realization of the permutations and associations would solve this problem in TGD framework.

4.4 Twistor Grassmannians and algebraic geometry

Twistor Grassmannians provide an application of algebraic geometry involving the above described notions [B2] (see <http://tinyurl.com/yd9tf2ya>). This approach allows extremely elegant expressions for planar amplitudes of $\mathcal{N} = 4$ SYM theory in terms of amplitudes formulated in Grassmannians $G(k, n)$.

It seems that this approach generalizes to TGD in such a way that CP_2 degrees of freedom give rise to additional factors in the amplitudes having form very similar to the M^4 part of amplitudes

and involving also $G(k, n)$ with ordinary twistor space CP_3 being replaced with the flag manifold $SU(3)/U(1) \times U(1)$: k would now correspond to the number of particles with negative weak isospin. Therefore the understanding of the algebraic geometry of twistor amplitudes could be helpful also in TGD framework.

4.4.1 Twistor Grassmannian approach very concisely

I try to compress my non-professional understanding of twistor Grassmann approach to some key points.

1. Twistor Grassmannian approach constructs the scattering amplitudes by fusing 3-vertices $(+, -, -)$ (one positive helicity) and $(-, +, +)$ (one negative helicity) to a more complex diagrams. All particles are on mass shell and massless but complex. If only real massless momenta are allowed the scattering amplitudes would allow only collinear gluons. Incoming particles have real momenta.

Remark: Remarkably, $M^4 \times CP_2$ twistor lift of TGD predicts also complex Noether charges, in particular momenta, already at classical level. Quantal Noether charges should be hermitian operators with real eigenvalues, which suggests that total Noether charges are real. For conformal weights this condition corresponds to conformal confinement. Also $M^8 - H$ duality requires a complexification of octonions by adding commuting imaginary unit and allows to circumvent problems related to the Minkowski signature since the metric tensor can be regarded as Euclidian metric tensor defining complex value norm as bilinear $m^k m_{kl} m^l$ in complexified M^8 so that real metric is obtained only in sub-spaces with real or purely imaginary coordinates. The additional imaginary unit allows also to define what complex algebraic numbers mean.

The unique property of 3-vertex is that the twistorial formulation for the conservation of four-momentum implies that in the vertex one has either $\lambda_1 \propto \lambda_2 \propto \lambda_3$ or $\bar{\lambda}_1 \propto \bar{\lambda}_2 \propto \bar{\lambda}_3$. These cases correspond to the 2 3-vertices distinguished notationally by the color of the vertex taken to be white or black [B2].

Remark: One must allow octonionic super-space in M^8 formulation so that octonionic SUSY broken by CP_2 geometry reducing to the quaternionicity of 8-momenta in given scattering diagram is obtained.

2. The conservation condition for the total four-momentum is quadratic in twistor variables for incoming particles. One can linearize this condition by introducing auxiliary Grassmannian $G(k, n)$ over which the tree amplitude can be expressed as a residue integral. The number theoretical beauty of the multiple residue integral is that it can make sense also p-adically unlike ordinary integral.

The outcome of residue integral is a sum of residues at discrete set of points. One can construct general planar diagrams containing loops from tree diagrams with loops by BCFW recursion. I have considered the possibility that BCFW recursion is trivial in TGD since coupling constants should be invariant under the addition of loops: the proposed scattering diagrammatics however assumed that scattering vertices reduce to scattering vertices for 2 fermions. The justification for renormalization group invariance would be number theoretical: there is no guarantee that infinite sum of diagrams gives simple function defined in all number fields with parameters in extension of rationals (say rational function).

3. The general form of the Grassmannian integrand in $G(k, n)$ can be deduced and follows from Yangian invariance meaning that one has conformal symmetries and their duals which expand to full infinite-dimensional Yangian symmetry. The denominator of the integrand of planar tree diagram is the product of determinants of $k \times k$ minors for the $k \times n$ matrix providing representation of a point of $G(k, n)$ unique apart from $SL(k, k)$ transformations. Only minors consisting of k consecutive columns are assumed in the product. The residue integral is determined by the poles of the denominator. There are also dynamical singularities allowing the amplitude to be non-vanishing only for some special configurations of the external momenta.

4. On mass-shell diagrams obtained by fusing 3-vertices are highly redundant. One can describe the general diagram by using a disk such that its boundary contains the external particles with positive or negative helicity. The diagram has certain number n_F of faces. There are moves, which do not affect the amplitude and it is possible to reduce the number of faces to minimal one: this gives what is called reduced diagram. Reduced diagrams with n_F faces define a unique $n_F - 1$ -dimensional sub-manifold of $G(k, n)$ over which the residue integral can be defined. Since the dimension of $G(k, n)$ is finite, also n_F is finite so that the number of diagrams is finite.
5. On mass shell diagrams can be labelled by the permutations of the external lines. This gives a connection with 1+1-dimensional QFTs and with braid group. In 1+1-D integral QFTs however scattering matrix induces only particle exchanges.

The permutation has simple geometric description: one starts from the boundary point of the diagram and moves always from left or right depending on the color of the point from which one started. One arrives some other point at the boundary and the final points are different for different starting points so that the process assigns a unique perturbation for a given diagram. Diagrams which are obtained by moves from each other define the same permutation. BFCW bridge which is a way to obtain new Yangian invariant corresponds to a permutation of consecutive external particles in the diagram.

6. The poles of the denominator determine the value of the multiple residue integrals. If one allowed all minors, one would have extremely complex structure of singularities. The allowance only cyclically taken minors simplifies the situation dramatically. Singularities correspond to n subgroups of more than 2 collinear k -vectors implying vanishing of some of the minors.
7. Algebraic geometry comes in rescue in the understanding of singularities. Since residue integral is in question, the choice is rather free and only the homology equivalence class of the cell decomposition matters. The poles for a hierarchy with poles inside poles since given singularity contains sub-singularities. This hierarchy gives rise to a what is known as cell composition - stratification - of Grassmannian consisting of varieties with various dimensions. These sub-varieties define representatives for the homology group of Grassmannian. Schubert cells already mentioned define this kind of stratification.

Remark: The stratification has very strong analogy of the decomposition of catastrophe in Thom's catastrophe theory to pieces of various dimensions. The smaller the dimension, the higher the criticality involved. A connection with quantum criticality of TGD is therefore highly suggestive.

Cyclicity implies a reduction of the stratification to that for positive Grassmannians for which the points are representable as $k \times n$ matrices with non-negative $k \times k$ determinants. This simplifies the situation even further.

Yangian symmetries have a geometric interpretation as symmetries of the stratification: level 1 Yangian symmetries are diffeomorphisms preserving the cell decomposition.

4.4.2 Problems of twistor approach

Twistor approach is extremely beautiful and elegant but has some problems.

1. The notion of twistor structure is problematic in curved space-times. In TGD framework the twistor structures of M^4 and CP_2 (E^4) induce twistor structure of space-time surface and the problem disappears just like the problems related to classical conservation laws are circumvented. Complexification of octonions allows to solve the problems related to the metric signature in twistorialization.
2. The description of massive particles is a problem. In TGD framework M^8 approach allows to replace massive particles with particles with octonionic momenta light-like in 8-D sense belonging to quaternionic subspace for a given diagram. The situation reduces to that for ordinary twistors in this quaternionic sub-space but since quaternionic sub-space can vary, additional degrees of freedom bringing in CP_2 emerge and manifest themselves as transversal 8-D mass giving real mass in 4-D sense.

3. Non-planar diagrams are also a problem. In TGD framework a natural guess is that they correspond to various permutations of free particle octonionic polynomials. Their product defines interaction region in the interior of CD to which free particles satisfying associativity conditions (quantum criticality) arrive. If the origins of polynomials are not along same time axis, the polynomials do not commute nor associate. One must sum over their permutations and for each permutation over its associations.

4.5 About the concrete construction of twistor amplitudes

At H -side the ground states of super-conformal representations are given by the anti-symmetrized products of the modes of H -spinor fields labelled by four-momentum, color quantum numbers, and electroweak (ew) quantum numbers. At partonic 2-surface one has finite number of many fermion states. Single fermion states are assigned with H -spinor basis and the fermion states form a representation of a finite-D Clifford algebra.

M^8 picture should reproduce the physical equivalent of H picture: in particular, one should understand four-momentum, color quantum numbers, ew quantum numbers, and B and L . $M^8 - H$ correspondence requires that the super-twistorial description of scattering amplitudes in M^8 is equivalent with that in H .

The M^8 picture is roughly following.

1. The ground states of super-conformal representations expressible in terms of spinor modes of H correspond at level of M^8 wave functions in super variant of the product $T(M^4) \times T(CP_2)$ of twistor spaces of M^4 and CP_2 . This twistor space emerges naturally in $M^8 - H$ correspondence from the quaternionicity condition for 8-momenta.
2. Bosonic M^8 degrees of freedom translate to wave functions in the product $T(M^4) \times T(CP_2)$ labelled by four-momentum and color. Super parts of the M^4 and CP_2 twistors code for spin and ew degrees of freedom and fermion numbers. Only a finite number of spin-ew spin states is possible for a given fundamental particle since one has finite-D Grassmann algebra.
3. Contrary to the earlier expectations [K18], the view about scattering diagrams is very similar to that in $\mathcal{N} = 4$ SUSY. The analog of 3-gluon vertex is fundamental and emerges naturally from number theoretic vision in which scattering diagrams defines a cognitive representation and vertices of the diagram correspond to fusion of sparticle lines.

4.5.1 Identification of H quantum numbers in terms of M^8 quantum numbers

The first challenge is to understand how $M^8 - H$ correspondence maps M^8 quantum numbers to H quantum numbers. At the level of M^8 one does not have action principle and conservation laws must follow from the properties of wave functions in various moduli spaces assignable to 4-D and 8-D CDs that is quaternion and octonion structures. The symmetries of the moduli spaces would dictate the properties of wave functions.

There are three types of symmetries and quantum numbers.

1. WCW quantum numbers

At level of H the quantum numbers in WCW “vibrational” degrees of freedom are associated with the representations of super-symplectic group acting as isometries of WCW. Super-symplectic generators correspond to Hamiltonians labelled by color and angular momentum quantum numbers for $SU(3) \times SO(3)$. In M^4_{\pm} there are also super-symplectic conformal weights assignable to the radial light-coordinate in δM^4_{\pm} . These conformal weights could be complex and might relate closely to the zeros of Riemann zeta [K13]. Physical states should however have integer valued conformal weights (conformal confinement).

At the level of M^8 WCW “vibrational” degrees of freedom are discrete and correspond to the degree of the octonionic polynomial P and its coefficients in the extension of rationals considered. WCW integration reduces to a discrete sum, which should be well-defined by the criticality conditions on the coefficients of the polynomials. $M^8 - H$ correspondence guarantees that 4-varieties in M^8 are mappable to space-time surfaces in H . Therefore also quantum numbers should be mappable to each other.

There are also spinorial degrees of freedom associated with WCW spinors with spin-like quantum numbers assignable to fermionic oscillator operators labelled by spin, ew quantum numbers, fermion numbers, and by super-symplectic conformal weights.

2. *Quantum numbers assignable to isometries of H .*

These quantum numbers are special assignable to the ground states of the representations of Kac-Moody algebras associated with light-like partonic orbits.

1. The isometry group of H consists of Poincare group and color group for CP_2 . M^8 isometries correspond to $8 - D$ Poincare group. Only G_2 respects given octonion structure and 8-D Lorentz transformations transform to each other different octonion structures. Quantum numbers consist of 8-momentum and analogs of spin and ew spin. $M^8 - H$ correspondence is non-trivial since one must map light-like quaternionic 8-momenta to 4-momenta and color quantum numbers.
2. There are quantum numbers assignable to cm spinor degrees of freedom. They correspond for both M^8 and H to 8-D spinors and give rise to spin and ew quantum numbers. For these quantum numbers $M^8 - H$ correspondence is trivial. At the level of H baryon and lepton numbers are assignable to the conserved chiralities of H -spinors.

Quantum classical correspondence (QCC) is a key piece of TGD.

1. At the level of H QCC states that the eigenvalues of the fermionic Noether charges are equal to the classical bosonic Noether charges in Cartan algebra implies that fermionic quantum number as also ew quantum numbers and spin have correlates at the level of space-time geometry.
2. At the level of M^8 QCC is very concrete. Both bosonic and superpart of octonions have the decomposition $1 + \bar{1} + 3 + \bar{3}$ under color rotations. Each monomial of theta parameters characterizes one particular many-fermion state containing leptons/antileptons and quarks/antiquarks. Leptons/antileptons are assignable to complexified octonionic units $(1 \pm iI_1)/\sqrt{2}$ defining preferred octonion plane M_2 and quarks/antiquarks are assignable to triplet and antitriplet, which also involve complexified octonion units. One obtains breaking of SUSY in the sense that space-time varieties assignable to different theta monomials are different (one can argue that the sum $8_s + \bar{8}_s$ can be regarded as real).

Purely leptonic and antileptonic varieties correspond to 1 and $\bar{1}$ and quark and antiquark varieties to 3 and $\bar{3}$ and the monomial transforms as a tensor product of thetas. The monomial has well defined quark and lepton numbers and the interpretation is that it characterizes fundamental sparticle. At the level of H this kind of correspondence follows from QCC.

3. Also super-momentum leads to a characterization of spin and fermion numbers of the state since delta function expressing conservation of super-momentum codes the supersymmetry for scattering amplitudes and gives rise to vertices conserving fermion numbers. Does this mean QCC in the sense that the super parts of super-momentum and super twistor should be associated with space-time varieties with same fermion and spin content?

How the light-like quaternionic 8-momenta are mapped to H quantum numbers?

The key challenge is to understand how the light-like quaternionic 8-momenta are mapped to massive M^4 momenta and color quantum numbers.

1. One has wave function in the space of CP_2 quaternionic four-momenta. M_0^4 momentum can be identified as M_0^2 projection and in general massive unless M_0^2 and M_0^4 are chosen so that the light-like M^8 momentum belongs to M_0^2 . The situation is analogous to that in the partonic description of hadron scattering.

The space of quaternionic sub-spaces $M_0^4 \supset M_0^2$ with this property is parameterized by CP_2 , and one obtains color partial waves. The inclusion of the choice of quantization axis extends this space to $T(CP_2) = SU(3)/U(1) \times U(1)$. Without quaternionicity/associativity condition the space of momenta would correspond to M^8 .

The wave functions in the moduli space for the position of the tip of CD and for the choice $M_0^2 \supset M_0^4$ specifying M_0^4 twistor structure and choice of quantization axis of spin correspond to wave functions in the twistor space CP_3 of M_{\pm}^4 coding for momentum and spin.

Remark: The inclusion of M^4 spin quantization axis characterized by the choice of M_0^2 extends M_0^4 to geometric twistor space $T(M^4) = M_0^4 \times S^2 \supset M_0^2$ having bundle projection to CP_3 . Twistorialization means essentially the inclusion of the choice of various quantization axis as degrees of freedom. This space is for symmetry group G the space G/H , where H is the Cartan sub-group of G . This description might make sense also at the level of super-symplectic and super-Kac-Moody symmetries.

2. Ordinary octonionic degrees of freedom for super-octonions in M^8 must be mapped to $M^4 \times CP_2$ cm degrees of freedom. Super octonionic parts should correspond to fermionic and spin and electroweak degrees of freedom. The space of super-twistorial states should same as the space of the super-symplectic grounds states describable in terms H -spinor modes.
3. One has wave function in the moduli space of CDs. The states in M^8 are labelled by quaternionic super-momenta. Bosonic part must correspond to four-momentum and color and super-part to spin and ew quantum numbers of CP_2 . This part of the moduli space wave function is characterized by the spin and ew spin quantum numbers of the fundamental particle. Wave functions in the super counterpart of $T(M^4) \times T(CP_2)$ allow to characterize these degrees of freedom without the introduction of spinors and should correspond to the ground states of super-conformal representations in H .

It seems that H -description is an abstract description at the level moduli spaces and M^8 description for single space-time variety represents reduction to the primary level, where number theory dictates the dynamics.

4.5.2 Octonionic twistors and super-twistors

How to define octonionic twistors? Or is it enough to identify quaternionic/associative twistors as sub-spaces of octonionic twistors?

1. Ordinary twistors and super-twistors

Consider first how ordinary twistors and their super counterparts could be defined, and how they could allow an elegant description of spin and ew quantum numbers as quantum numbers analogous to angular momenta.

1. Ordinary twistors are defined as pairs of 2-spinors giving rise to a representation of four-momentum. The spinors are complex spinors transforming as a doublet representation of $SL(2, \mathbb{C})$ and its conjugate.

The 2-spinors are related by incidence relation, a linear condition in which M^4 coordinates represented as 2×2 matrix appears linearly [K18]. The expression of four-momentum is bilinear in the spinors and invariant under complex scalings of the 2-spinors compensating each other so that instead of 8-D space one has actually 6-D space, which reduces to CP_3 to which the geometric twistor space $M^4 \times S^2$ has a projection.

2. For light-like four-momenta p the determinant of the matrix having the two 2-spinors as rows and representing p as a point of M^4 vanishes. Wave functions in CP_3 allow to describe spin in terms of bosonic wave function. What is so beautiful is that this puts particles with different spin in a democratic position.

Super-twistors allow to integrate the states constructible as many-fermion states of \mathcal{N} elementary fermions in the same representations involving several spins. The many-fermion states - sparticles - are in 1-1-correspondence with Grassmann algebra basis.

3. The description of massless particles in terms of M^4 (super-)twistors is elegant but one encounters problems in the case of massive particles [K21, K15, K18].

2. *Octonionic twistors at the level of M^8 ?*

How to define octonionic twistors at the level of M^8 ?

1. At the level of M^8 one has light-like 8-momenta. The M^4 momentum identified as M_0^4 projection can there be massive. This solves the basic problem of the standard twistor approach.
2. The additional assumption is that the 8-momenta in given vertex of scattering diagram belong to the same quaternionic sub-space $M_0^4 \subset M^8$ satisfying $M_0^4 \supset M_0^2$. This effectively transforms momentum space $M^4 \times E^4$ to $M^4 \times CP_2$. A stronger condition is that all momenta in a given diagram belong to the same sub-space $M_0^4 \supset M_0^2$.

Remark: Quaternionicity implies that the 8-momentum is time-like or light-like if one requires that quaternionicity for an arbitrary choice of the octonionic structure (the action of 8-D Poincare group gives rise transforms octonionic structures to each other).

3. Complex 2-spinors are replaced with complexified octonionic spinors which must be consistent quaternionicity condition for 8-momenta. A good guess is that the spinors belong to a quaternionic sub-space of octonions too. This is expected to transform them effectively to quaternionic spinors. Without effective quaternionicity the number of 2-spinor components would be 8 rather than 4 times larger than for ordinary 2-spinors.

Remark: One has complexified octonions (i commutes with the octonionic imaginary units E_k).

4. Octonionic/quaternionic twistors should be pairs of octonionic/quaternionic 2-spinors determined only modulo octonionic/quaternionic scaling. If quaternionicity holds true, the number of 2-spinor components is 4 times larger than usually. Does this mean that one has basically quaternionic twistors plus moduli space CP_2 for $M_0^4 \supset M_0^2$. One should be able to express octonionic twistors as bi-linears formed from 2 octonionic/quaternionic 2-spinors. Octonionic option should give the octonionic counterpart OP_3 of Grassmannian CP_3 , which does not however exist.

Remark: Octonions allow only projective plane OP_2 as the octonionic counterpart of CP_2 (see <http://tinyurl.com/ybwaeu2s>) but do not allow higher-D projective spaces nor Grassmannians (see <http://tinyurl.com/ybm8ubef>, whereas reals, complex numbers, and quaternions do so. The non-existence of Grassmannians for rings obtained by Cayley-Dickson construction could mean that $M^8 - H$ correspondence and TGD do not generalize beyond octonions.

Does the restriction to quaternionic 8-momenta the Grassmannians to be quaternionic (sub-spaces of octonions). This would give quaternionic counterpart HP_3 of CP_3 . Quaternions indeed allow projective spaces and Grassmannians and (see <http://tinyurl.com/y9htjstc> and <http://tinyurl.com/y87gpq81>).

Remark: One can wonder whether non-commutativity forces to distinguish between left- and right Grassmannians (points as lines $\{c(q_1, \dots, q_n) | c \in H\}$ or as lines as lines $\{(q_1, \dots, q_n)c | c \in H\}$).

5. Concerning the generalization to octonionic case, it is crucial to realize that the 2×2 -matrix representing four-momentum as a pair 2-spinor can be regarded as an element in the sub-space of complexified quaternions. The representation of four-momentum would be as sum of $p_8 = p_1^k \sigma_k + I_4 p_2^k \sigma_k$, where I_4 octonionic imaginary unit orthogonal to σ_k representing quaternionic units.

No! The twistorial representation of the 4-momentum is already quaternionic! Choosing the decomposition of M^8 to quaternionic sub-space and its complement suitably, one has $IM(p_8) = 0$ for quaternionic 8-momenta and one obtains standard representation of 4-momentum in this sub-space! The only new element is that one has now moduli specifying the quaternionic sub-space. If the sub-space contains a fixed M_0^2 one obtains just CP_2 and ordinary twistor codes for the choices of M_0^2 . If the choice of color quantization axes matters

as it indeed does, one has twistor space $SU(3)/U(1) \times U(1)$ instead of CP_2 . This would suggest that ordinary representation of scattering amplitudes reduces apart from the presence of CP_2 twistor to the usual representation.

One can hope for a reduction to ordinary twistors and projective spaces, moduli space CP_2 for quaternion structures, and moduli space for the choices of real axis of octonion structures. One can even consider the possibility [K18] of using standard M_0^2 with the property that M^8 momentum reduces to M_0^2 momentum and coding the information about real M_0^2 to moduli. This could reduce the twistor space to $RP(3)$ associated with M_0^2 is considered and solve the problems related to the signature of M^4 . Note however that the complexification of octonions in any case allow to regard the metric as Euclidian albeit complexified so that these problems should disappear.

3. Octonionic super-twistors at the level of M^8 ?

Should one generalize the notion of super-twistor to octonionic context or can one do by using only the moduli space and the fact that octonionic geometry codes for various components of octonion as analog of super-field? It seems that super-twistors are needed.

1. It seems that super-twistors are needed. Octonionic super-momentum would appear in the super variant of momentum conserving delta function resulting in the integration over translational moduli. In twistor Grassmann approach this delta function is super-twistorialized and this leads to the amazingly simple expressions for the scattering amplitudes.
2. At the level of M^8 one should generalize ordinary momentum to super-momentum and perform super-twistorialization. Different monomials of theta parameters emerging from super part of momentum conserving delta function (for $\mathcal{N} = 1$ one has $\delta(\theta - \theta_0) = \exp(i\theta - \theta_0)/i$) correspond to different spin states of the super multiplet and anti-commutativity guarantees correct statistics. At the level of H the finite-D Clifford algebra of 8-spinors at fixed point of H gives states obtained as monomials or polynomials for the components of super-momentum in M^8 .
3. Octonionic super-momentum satisfying quaternionicity condition can be defined as a combination of ordinary octonionic 8-momentum and super-parts transforming like 8_s and $\bar{8}_s$. One can express the octonionic super-momentum as a bilinear of the super-spinors defining quaternionic super-twistor. Quaternionicity is assumed at least for the octonionic super-momenta in the same vertex. Hence the M^4 part of the super-twistorialization reduces to that in SUSYs and one obtains standard formulas. The new elements is the super-twistorialization of $T(CP_2)$.

Remark: Octonionic SUSY involving $8 + 8_s + \bar{8}_s$ would be an analog of $\mathcal{N} = 8$ SUSY associated with maximal supergravity (see <http://tinyurl.com/nv3aaajy>) and in M^4 degrees of freedom twistorialization should be straightforward.

The octonionic super-momentum belongs to a quaternionic sub-space labelled by CP_2 point and corresponds to a particular sub-space M_0^2 in which it is light-like (has no other octonionic components). M_0^2 is characterized by point of S^2 point of twistor space $M^4 \times S^2$ having bundle projection to CP_3 .

4. That the twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$ coding for the color quantization axes rather than only CP_2 emerges must relate to the presence of electroweak quantum numbers related to the super part of octonionic momentum. Why the rotations of $SU(2) \times U(1) \subset SU(3)$ have indeed interpretation also as tangent space-rotations interpreted as electroweak rotations. The transformations having an effect on the choice of quantization axes are parameterized by S^2 relating naturally to the choice of $SO(4)$ quantization axis in E^4 and coded by the geometric twistor space $T(E^4) = E^4 \times S^2$.
5. Since the super-structure is very closely related to the construction of the exterior algebra in the tangent space, super-twistorialization of $T(CP_2)$ should be possible. Octonionic triality could be also in a key role and octonionic structure in the tangent space of $SU(3)$ is highly suggestive. $SU(3)$ triality could relate to the octonionic triality.

$SU(3)/U(1) \times U(1)$ is analogous with the ordinary twistor space CP_3 obtained from C^4 as a projective space. Now however $U(1) \times U(1)$ instead of group of complex scalings would define the equivalence classes. Generalization of projective space would be in question. The superpart of twistor would be obtained as $U(1) \times U(1)$ equivalence class and gauge choice should be possible to get manifestly 6-D representation. One can ask whether the CP_2 counterparts of higher- D Grassmannians appear at the level of generalized twistor diagrams: could the spaces $SU(n)/G$, H Cartan group correspond to these spaces?

4. How the wave functions in super-counterpart of $T(CP_2)$ correspond to quantum states in CP_2 degrees of freedom?

In CP_2 spinor partial waves have vanishing triality $t = 0$ for leptonic chirality and $t = \pm 1$ for quarks and antiquarks. One can say that the triality $t \neq 1$ states are possible thanks to the anomalous hypercharge equal to fractional electromagnetic charge $Y_A = Q_{em}$ of quarks: this gives also correlation between color quantum numbers and electroweak quantum numbers which is wrong for spinor partial waves. The super-symplectic and super Kac-Moody algebras however bring in vibrational degrees of freedom and one obtains correct quantum number assignments [K4].

This mechanism should have a counterpart at the level of the super variant of the twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$. The group algebra of $SU(3)$ gives the scalar wave functions for all irreps of $SU(3)$ as matrix elements. Allowing only matrix elements that are left- or right invariant under $U(1) \times U(1)$ one obtains all irreps realized in $T(CP_2)$ as scalar wave functions. These representations have $t = 0$. The situation would be analogous for scalar functions in CP_2 . One must however obtain also electroweak quantum numbers and $t \neq 0$ colored states. Here the octonionic algebraic geometry and superpart of the $T(CP_2)$ should come in rescue. The electroweak degrees of freedom in CP_2 should correspond to the super-parts of twistors.

The $SU(3)$ triplets assignable to the triplets 3 and $\bar{3}$ of space-time surfaces would make possible also the $t = \pm 1$ states. Color would be associated with the octonionic geometry. The simplest possibility would be that one has just tensor products of the triplets with $SU(3)/U(1) \times U(1)$ partial waves. In the case of CP_2 there is however a correlation between color partial waves and electroweak quantum numbers and the same is expected also now between super-part of the twistor and geometric color wave function: minimum correlation is via $Y_A = Q_{em}$. The minimal option is that the number theoretic color for the octonionic variety modifies the transformation properties of $T(CP_2)$ wave function only by a phase factor due to $Y_A = Q_{em}$ as in the case of CP_2 .

The most elegant outcome would be that super-twistorial state basis in $T(M^4) \text{ times } T(CP_2)$ is equivalent with the state basis defined by super-symplectic and super Kac-Moody representations in H .

4.5.3 About the analogs of twistor diagrams

There seems to be a strong analogy with the construction of twistor amplitudes in $\mathcal{N} = 4$ SUSY [B1, B4, B3] and one can hope of obtaining a purely geometric analog of SUSY with dynamics of fields replaced by the dynamics of algebraic super-octonionic surfaces.

1. Number theoretical vision leads to the proposal that the scattering amplitudes involve only data at discrete points of the space-time variety belonging to extension of rationals defining cognitive representation. The identification of these points has been already considered in the case of partonic orbits entering to the partonic 2-vertex and for the regions of space-time surfaces intersecting at discrete set of points. Scattering diagrams should therefore correspond to polygons with vertices of polygons defining cognitive representation and lines assignable to the external fundamental particles with given quark and lepton numbers having correlates at the level of space-time geometry. This occurs also in twistor Grassmannian approach [B1, B4, B3].

Since polynomials determine space-time surfaces, this data is enough to determine the space-time variety completely. Indeed, the zeros of $P(t)$ determining the space-time variety give also rise to a set of spheres $S^6(t_n)$ and partonic 2-surfaces $X^2(t_n) = X^4 \cap S^6(t_n)$, where t_n is root of $P(t)$. The discretization need not mean a loss of information. The scattering amplitudes would be expressible as an analog of n -point function with points having coordinates in the extension of rationals.

2. (Super) octonion as “field” in X^4 is dynamically analogous to (super) gauge potentials and super-octonion to its super variant. (Super) gauge potentials are replaced with M^8 (super-) octonion coordinate and gauge interactions are geometrized. Here I encounter a problem with terminology. Neither sparticle nor sboson sounds good. Hence I will talk about sparticles.
3. The amplitude for a given space-time variety contains no information M^8 -momentum. M^8 -momentum emerges as a label for a wave function in the moduli space of 4-D and 8-D CDs involving both translational and orientational degrees of freedom. For fixed time axis the orientational degrees of freedom reduce to rotational degrees of freedom identifiable in terms of the twistor sphere S^2 . The delta functions expressing conservation of 8-D quaternionic super-momentum in M^8 coming from the integration over the moduli space of 8-D translations.

As found, quaternionicity of 8-momenta implies that standard M^4 twistor description of momenta applies but one obtains CP_2 twistors as additional contribution. This is of course what one would intuitively expect.

8-D momentum conservation in turn translates to the conservation of momentum and color quantum numbers in the manner described. The amplitudes in momentum and color degrees of freedom reduce to kinematics as in SUSYs. It is however not clear whether one should also perform number theoretical discretization of various moduli spaces.

In any case, it seems that all the details of the scattering amplitudes related to moduli spaces reduce to symmetries and the core of calculations reduces to the construction of space-time varieties as zero loci of octonionic polynomials and identification of the points of the 4-varieties in extension of rationals. Classical theory would indeed be an exact part of the quantum theory.

4. Quaternionic 8-D light-likeness reduces the situation to the level of ordinary complex and thus even positive (real) Grassmannians. This is crucial from the p-adic point of view. CP_2 twistors characterizes the moduli related to the choice of quaternionic sub-space, where 8-momentum reduces to ordinary 4-momentum. M^4 parts of the scattering amplitudes in twistor Grassmann approach should be essentially the same as in $\mathcal{N} = 4$ SUSY apart from the replacement of super degrees of freedom with super-octonionic ones. The challenge is to generalize the formalism so that it applies also to CP_2 twistors. The challenge would be to generalize the formalism so that it applies also to CP_2 twistors. The M^4 and CP_2 degrees of freedom are expected to factorize in twistorial amplitudes. A good guess is that the scattering amplitudes are obtained as residue integrals in the analogs of Grassmannians associated with $T(CP_2)$. Could one have Grassmannians also now?

Consider the formula of tree amplitude for n gluons with k negative helicities conjectured Arkani-Hamed *et al* in the twistor Grassmannian approach [B3]. The amplitude follows from the twistorial representation for momentum conservation and is equal to an $k \times n$ -fold multiple residue integral over the complex variables $C_{\alpha a}$ defining coordinates for Grassmannian $Gl(n, k)$ and reduces to a sum over residues. The integrand is the inverse for the product of all $k \times k$ minors of the matrix $C_{\alpha a}$ in cyclic order and the residues corresponds to zeros for one or more minors. This part does not depend on twistor variables. The dependence on n twistor variables comes from the product $\prod_{\alpha=1}^k \delta(C_{\alpha a} W^a)$ of k delta functions related to momentum conservation. W^a denotes super-twistors in the 8-D representation, which is linear. One has projective invariance and therefore a reduction to $T(M^4) = CP_3 = SU(4)/SU(3) \times U(1)$.

Could this formula generalize almost as such to $T(CP_2)$ and come from the conservation of E^4 momentum? One has n sparticles to which super-twistors in $T(CP_2)$ are assigned. The first guess is that the sign of helicity are replaced by the sign of electroweak isospin - essentially E^4 spin at the level of M^8 . For electromagnetic charge identified as the analog of helicity one would have problems in the case of neutrinos. $T(M^4) = CP_3 = SU(4)/SU(3) \times U(1)$ is replaced with $T(CP_2) = SU(3)/U(1) \times U(1)$. $T(CP_2)$ does not have a representation as a projective space but there is a close analogy since the group of complex scalings is replaced with $U(1) \times U(1)$. The (apparent) linearity is lost but one represent the points of $T(CP_2)$ as exponentials of $su(3)$ Lie-algebra elements with vanishing $u(1) \times u(1)$ part. The resulting 3 complex coordinates are analogous to two complex CP_2 coordinates. The basic difference

between M^4 and CP_2 degrees of freedom would come from the exponential representation of twistors.

5. By Yangian invariance one should obtain very similar formulas for the amplitudes except that one has instead of $\mathcal{N} = 4$ SUSY $\mathcal{N} = 8$ octonionic SUSY analogous to $\mathcal{N} = 8$ SUGRA.

4.5.4 Trying to understand the fundamental 3-vertex

Due to its unique twistorial properties as far as realization of four-momentum conservation is considered 3-vertex is fundamental in the construction of scattering diagrams in twistor Grassmannian approach to $\mathcal{N} = 4$ SYM [B2] (see <http://tinyurl.com/yd9tf2ya>). Twistor Grassmann approach suggests that 3-vertex with complexified light-like 8-momenta represents the basic building brick representing from which more complex diagrams can be constructed using the BCFW recursion formula [B2]. In TGD 3-vertex generalized to 8-D light-like quaternionic momenta should be highly analogous to the 4-D 3-vertex and in a well-defined sense reduce to it if all momenta of the diagram belong to the same quaternionic sub-space M_0^4 . It is however not completely clear how 3-vertex emerges in TGD framework.

1. A possible identification of the 3-vertex at the level of M^8 would be as a vertex at which 3 sparticle lines with light-like complexified quaternionic 8-momenta meet. This vertex would be associated with the partonic vertex $X^2(t_n) = X^4 \cap S^6(t_n)$. Incoming sparticle lines at the light-like partonic orbits identified as boundaries of string world sheets (for entangled states at least) would be light-like.

Does the fusion of two sparticle lines to third one require that either or both fusing lines become space-like - say pieces of geodesic line inside the Euclidian space-time region- bounded by the partonic orbit? The identification of the lines of twistor diagrams as carriers of light-like complexified quaternionic momenta in 8-D sense does not encourage this interpretation (also classical momenta are complex). Should one pose the fusion of the light-like lines as a boundary condition? Or should one give up the idea that sparticle lines make sense inside interaction region?

2. As found, one can challenge the assumption about the existence of string world sheets as commutative regions in the non-associative interaction region. Could one have just fermion lines as light-like curves at partonic orbits inside CD? Or cannot one have even them?

Even if the polynomial $\prod_i P_i$ defining the interaction region is product of polynomials with origins of octonionic coordinates not along the same real line, the 7-D light-cones of M^8 associated with the particles still make sense in the sense that $P_i(o_i) = 0$ reduces at it to $P_i(t_i) = 0$, t_i real number, giving spheres $S^6(t_i(n))$ and partonic 2-surfaces and vertices $X_2(t_i(n))$. The light-like curves as geodesics the boundary of 7-D light-cones mapped to light-like curves along partonic orbits in H would not be lost inside interaction regions.

3. At the level of H this relates to a long standing interpretational problem related to the notion of induced spinor fields. SH suggests strongly the localization of the induced spinor fields at string world sheets and even at sparton lines in absence of entanglement. Super-conformal symmetry however requires that induced spinor fields are 4-D and thus seems to favor delocalization. The information theoretic interpretation is that the induced spinor fields at string world sheets or even at sparton lines contain all information needed to construct the scattering amplitudes. One can also say that string world sheets and sparton lines correspond to a description in terms of an effective action.

4.5.5 Could the M^8 view about twistorial scattering amplitudes be consistent with the earlier H picture?

The proposed M^8 picture involving super coordinates of M^8 and super-twistors does not conform with the earlier proposal for the construction of scattering amplitudes at the level of H [K18]. In H picture the introduction of super-space does not look natural, and one can say that fundamental fermions are the only fundamental particles [K15, K18]. The H view about super-symmetry is as

broken supersymmetry in which many fermion states at partonic 2-surfaces give rise to supermultiplets such that fermions are at different points. Fermion 4-vertex would be the fundamental vertex and involve classical scattering without fusion of fermion lines. Only a redistribution of fermion and anti-fermion lines among the orbits of partonic 2-surfaces would take place in scattering and one would have kind of OZI rule.

Could this H view conform with the recent M^8 view much closer to the SUSY picture. The intuitive idea without a rigorous justification has been that the fermion lines at partonic 2-surfaces correspond to singularities of many-sheeted space-time surface at which some sheets co-incide. M^8 sparticle consists effectively of n fermions at the same point in M^8 . Could it be mapped by $M^8 - H$ duality to n fermions at distinct locations of partonic 2-surface in H ?

$M^8 - H$ correspondence maps the points of $M^4 \subset M^4 \times E^4$ to points of $M^4 \subset M^4 \times CP_2$. The tangent plane of space-time surface containing a preferred M^2 is mapped to a point of CP_2 . If the effective n -fermion state M^8 is at point at which n sheets of space-time surface co-incide and if the tangent spaces of different sheets are not identical, which is quite possible and even plausible, the point is indeed mapped to n points of H with same M^4 coordinates but different CP_2 coordinates and sparticle would be mapped to a genuine many-fermion state. But what happens to scalar sparticle. Should one regard it as a pure gauge degree of freedom in accordance with the chiral symmetry at the level of M^8 and H ?

5 From amplituhedron to associahedron

Lubos has a nice blog posting (see <http://tinyurl.com/y7ywhxew>) explaining the proposal represented in the newest article by Nima Arkani-Hamed, Yuntao Bai, Song He, Gongwang Yan [?]see <http://tinyurl.com/ya8zst11>). Amplituhedron is generalized to a purely combinatorial notion of associahedron and shown to make sense also in string theory context (particular bracketing). The hope is that the generalization of amplituhedron to associahedron allows to compute also the contributions of non-planar diagrams to the scattering amplitudes - at least in $\mathcal{N} = 4$ SYM. Also the proposal is made that color corresponds to something less trivial than Chan-Paton factors.

The remaining problem is that 4-D conformal invariance requires massless particles and TGD allows to overcome this problem by using a generalization of the notion of twistor: masslessness is realized in 8-D sense and particles massless in 8-D sense can be massive in 4-D sense.

In TGD non-associativity at the level of arguments of scattering amplitude corresponds to that for octonions: one can assign to space-time surfaces octonionic polynomials and induce arithmetic operations for space-time surface from those for polynomials (or even rational or analytic functions). I have already earlier [L6] demonstrated that associahedron and construction of scattering amplitudes by summing over different permutations and associations of external particles (space-time surfaces). Therefore the notion of associahedron makes sense also in TGD framework and summation reduces to “integration” over the faces of associahedron. TGD thus provides a concrete interpretation for the associations and permutations at the level of space-time geometry.

In TGD framework the description of color and four-momentum is unified at the level and the notion of twistor generalizes: one has twistors in 8-D space-time instead of twistors in 4-D space-time so Chan-Paton factors are replaced with something non-trivial.

5.1 Associahedrons and scattering amplitudes

The following describes briefly the basic idea between associahedrons.

5.1.1 Permutations and associations

One starts from a non-commutative and non-associative algebra with product (in TGD framework this algebra is formed by octonionic polynomials with real coefficients defining space-time surfaces as the zero loci of their real or imaginary parts in quaternionic sense. One can indeed multiply space-time surface by multiplying corresponding polynomials! Also sum is possible. If one allows rational functions also division becomes possible.

All permutations of the product of n elements are in principle different. This is due to non-commutativity. All associations for a given ordering obtained by scattering bracket pairs in the product are also different in general. In the simplest case one has either $a(bc)$ or $(ab)c$ and these

2 give different outcomes. These primitive associations are building bricks of general associations: for instance, abc does not have well-defined meaning in non-associative case.

If the product contains n factors, one can proceed recursively to build all associations allowed by it. Decompose the n factors to groups of m and $n - m$ factors. Continue by decomposing these two groups to two groups and repeat until you have groups consisting of 1 or two elements. You get a large number of associations and you can write a computer code computing recursively the number $N(n)$ of associations for n letters.

Two examples help to understand. For $n = 3$ letters one obviously has $N(n = 3) = 2$. For $n = 4$ one has $N(4) = 5$: decompose first $abcd$ to $(abc)d$, $a(bcd)$ and $(ab)(cd)$ and then the two 3 letter groups to two groups: this gives $N(4) = 2 + 2 + 1 = 5$ associations and associahedron in 3-D space has therefore 5 faces.

5.1.2 Geometric representation of association as face of associahedron

Associations of n letters can be represented geometrically as so called Stasheff polytope (see <http://tinyurl.com/q9ga785>). The idea is that each association of n letters corresponds to a face of polytope in $n - 2$ -dimensional space with faces represented by the associations.

Associahedron is constructed by using the condition that adjacent faces (now 2-D polygons) intersecting along common face (now 1-D edges). The number of edges of the face codes for the structure particular association. Neighboring faces are obtained by doing minimal change which means replacement of some $(ab)c$ with $a(bc)$ appearing in the association as a building bricks or vice versa. This means that the changes are carried out at the root level.

5.1.3 How does this relate to particle physics?

In scattering amplitude letters correspond to external particles. Scattering amplitude must be invariant under permutations and associations of the external particles. In particular, this means that one sums over all associations by assigning an amplitude to each association. Geometrically this means that one "integrates" over the boundary of associahedron by assigning to each face an amplitude. This leads to the notion of associahedron generalizing that of amplituhedron.

Personally I find it difficult to believe that the mere combinatorial structure leading to associahedron would fix the theory completely. It is however clear that it poses very strong conditions on the structure of scattering amplitudes. Especially so if the scattering amplitudes are defined in terms of "volumes" of the polyhedrons involved so that the scattering amplitude has singularities at the faces of associahedron.

An important constraint on the scattering amplitudes is the realization of the Yangian generalization of conformal symmetries of Minkowski space. The representation of the scattering amplitudes utilizing moduli spaces (projective spaces of various dimensions) and associahedron indeed allows Yangian symmetries as diffeomorphisms of associahedron respecting the positivity constraint. The hope is that the generalization of amplituhedron to associahedron allows to generalize the construction of scattering amplitudes to include also the contribution of non-planar diagrams of at $\mathcal{N} = 4$ SYM in QFT framework.

5.2 Associations and permutations in TGD framework

Also in the number theoretical vision about quantum TGD one encounters associativity constraints leading to the notion of associahedron. This is closely related to the generalization of twistor approach to TGD forcing to introduce 8-D analogs of twistors [L6] (see <http://tinyurl.com/yd43o2n2>).

5.2.1 Non-associativity is induced by octonic non-associativity

As found in [L6], non-associativity at the level of space-time geometry and at the level of scattering amplitudes is induced from octonionic non-associativity in M^8 .

1. By $M^8 - H$ duality ($H = M^4 \times CP_2$) the scattering are assignable to complexified 4-surfaces in complexified M^8 . Complexified M^8 is obtained by adding imaginary unit i commuting with octonionic units I_k , $k = 1, \dots, 7$. Real space-time surfaces are obtained as restrictions

to a Minkowskian subspace complexified M^8 in which the complexified metric reduces to real valued 8-D Minkowski metric. This allows to define notions like Kähler structure in Minkowskian signature and the notion of Wick rotations ceases to be ad hoc concept. Without complexification one does not obtain algebraic geometry allowing to reduce the dynamics defined by partial differential equations for preferred extremals in H to purely algebraic conditions in M^8 . This means huge simplifications but the simplicity is lost at the QFT-GRT limit when many-sheeted space-time is replaced with slightly curved piece of M^4 .

2. The real 4-surface is determined by a vanishing condition for the real or imaginary part of octonionic polynomial with $RE(P)$ and $IM(P)$ defined by the composition of octonion to two quaternions: $o = RE(o) + I_4 IM(o)$, where I_4 is octonionic unit orthogonal to a quaternionic sub-space and $RE(o)$ and $IM(o)$ are quaternions. The coefficients of the polynomials are assumed to be real. The products of octonionic polynomials are also octonionic polynomials (this holds for also for general power series with real coefficients (no dependence on I_k). The product is not however neither commutative nor associative without additional conditions. Permutations and their associations define different space-time surfaces. The exchange of particles changes space-time surface. Even associations do it. Both non-commutativity and non-associativity have a geometric meaning at the level of space-time geometry!
3. For space-time surfaces representing external particles associativity is assumed to hold true: this in fact guarantees $M^8 - H$ correspondence for them! For interaction regions associativity does not hold true but the field equations and preferred extremal property allow to construct the counterpart of space-time surface in H from the boundary data at the boundaries of CD fixing the ends of space-time surface.

Associativity poses quantization conditions on the coefficients of the polynomial determining it. The conditions are interpreted in terms of quantum criticality. In the interaction region identified naturally as causal diamond (CD), associativity does not hold true. For instance, if external particles as space-time surfaces correspond to vanishing of $RE(P_i)$ for polynomials representing particles labelled by i , the interaction region (CD) could correspond to the vanishing of $IM(P_i)$ and associativity would fail. At the level of H associativity and criticality corresponds to minimal surface property so that quantum criticality corresponds to universal free particle dynamics having no dependence on coupling constants.

4. Scattering amplitudes must be commutative and associative with respect to their arguments which are now external particles represented by polynomials P_i . This requires that scattering amplitude is sum over amplitudes assignable to 4-surfaces obtained by allowing all permutations and all associations of a given permutation. Associations can be described combinatorially by the associahedron!

Remark: In quantum theory associative statistics allowing associations to be represented by phase factors can be considered (this would be associative analog of Fermi statistics). Even a generalization of braid statistics can be considered.

Yangian variants of various symmetries are a central piece also in TGD although supersymmetries are realized in different manner and generalized to super-conformal symmetries: these include generalization of super-conformal symmetries by replacing 2-D surfaces with light-like 3-surfaces, supersymplectic symmetries and dynamical Kac-Moody symmetries serving as remnants of these symmetries after supersymplectic gauge conditions characterizing preferred extremals are applied, and Kac-Moody symmetries associated with the isometries of H . The representation of Yangian symmetries as diffeomorphisms of the associahedron respecting positivity constraint encourages to think that associahedron is a useful auxiliary tool also in TGD.

5.2.2 Is color something more than Chan-Paton factors?

Nima *et al* talk also about color structure of the scattering amplitudes usually regarded as trivial. It is claimed that this is actually not the case and that there is non-trivial dynamics involved. This is indeed the case in TGD framework. Also color quantum numbers are twistorialized in terms of the twistor space of CP_2 , and one performs a twistorialization at the level of M^8 and $M^4 \times CP_2$. At the level of M^8 momenta and color quantum numbers correspond to associative 8-momenta.

Massless particles are now massless in 8-D sense but can be massive in 4-D sense. This solves one of the basic difficulty of the ordinary twistor approach. A further bonus is that the choice of the embedding space H becomes unique: only the twistor spaces of S^4 (and generalized twistor space of M^4 and CP_2 have Kähler structure playing a crucial role in the twistorialization of TGD. To sum up, all roads lead to Rome. Everyone is well-come to Rome!

5.3 Questions inspired by quantum associations

Associations have (or seem to have) different meaning depending on whether one is talking about cognition or mathematics. In mathematics the associations correspond to different bracketings of mathematical expressions involving symbols denoting mathematical objects and operations between them. The meaning of the expression - in the case that it has meaning - depends on the bracketing of the expression. For instance, one has $a(b+c) \neq (ab)+c$, that is $ab+ac \neq ab+c$. Note that one can change the order of bracket and operation but not that of bracket and object.

For ordinary product and sum of real numbers one has associativity: $a(bc) = (ab)c$ and $a+(b+c) = (a+b)+c$. Most algebraic operations such as group product are associative. Associativity of product holds true for reals, complex numbers, and quaternions but not for octonions and this would be fundamental in both classical and quantum TGD.

The building of different associations means different groupings of n objects. This can be done recursively. Divide first the objects to two groups, divide these two groups to two groups each, and continue until you have division of 3 objects to two groups - that is abc divided into $(ab)c$ or $a(bc)$. Numbers 3 and 2 are clearly the magic numbers.

This inspire several speculative questions related to the twistorial construction of scattering amplitudes as associative singlets, the general structure of quantum entanglement, quantum measurement cascade as formation of association, the associative structure of many-sheeted space-time as a kind of linguistic structure, spin glass as a strongly associative system, and even the tendency of social structures to form associations leading from a fully democratic paradise to cliques of cliques of

1. In standard twistor approach 3-gluon amplitude is the fundamental building brick of twistor amplitudes constructed from on-shell-amplitudes with complex momenta recursively. Also in TGD proposal this holds true. This would naturally follow from the fact that associations can be reduced recursively to those of 3 objects. 2- and 3-vertex would correspond to a fundamental associations. The association defined 2-particle pairing (both associated particles having either positive or negative helicities for twistor amplitudes) and 3-vertex would have universal structure although the states would be in general decompose to associations.
2. Consider first the space-time picture about scattering [L6]. CD defines interaction region for scattering amplitudes. External particles entering or leaving CD correspond to associative space-time surfaces in the sense that the tangent space or normal space for these space-time surfaces is associative. This gives rise to $M^8 - H$ correspondence.

These surfaces correspond to zero loci for the imaginary parts (in quaternionic sense) for octonionic polynomial with coefficients, which are real in octonionic sense. The product of $\prod_i P_i$ of polynomials with same octonion structure satisfying $IM(P_i) = 0$ has also vanishing imaginary part and space-time surface corresponds to a disjoint union of surfaces associated with factors so that these states can be said to be non-interacting.

Neither the choice of quaternion structure nor the choice of the direction of time axis assignable to the octonionic real unit need be same for external particles: if it is the particles correspond to same external particle. This requires that one treats the space of external particles (4-surfaces) as a Cartesian product of of single particle 4-surfaces as in ordinary scattering theory.

Space-time surfaces inside CD are non-associative in the sense that the neither normal nor tangent space is associative: $M^8 - M^4 \times CP_2$ correspondence fails and space-time surfaces inside CD must be constructed by applying boundary conditions defining preferred extremals. Now the real part of $RE(\prod_i P_i)$ in quaternionic sense vanishes: there is genuine interaction even when the incoming particles correspond to the same octonion structure since one does not have union of surfaces with vanishing $RE(P_i)$. This follows from a rather trivial

observation holding true already for complex numbers: imaginary part of zw vanishes if it vanishes for z and w but this does not hold true for the real part. If octonionic structures are different, the interaction is present irrespective of whether one assumes $RE(\prod_i P_i) = 0$ or $IM(\prod_i P_i) = 0$. $RE(\prod_i P_i) = 0$ is favoured since for $IM(\prod_i P_i) = 0$ one would obtain solutions for which $IM(P_i) = 0$ would vanish for the i :th particle: the scattering dynamics would select i :th particle as non-interacting one.

3. The proposal is that the entire scattering amplitude defined by the zero energy state - is associative, perhaps in the projective sense meaning that the amplitudes related to different associations relate by a phase factor (recall that complexified octonions are considered), which could be even octonionic. This would be achieved by summing over all possible associations.
4. Quantum classical correspondence (QCC) suggests that in ZEO the zero energy states - that is scattering amplitudes determined by the classically non-associative dynamics inside CD - form a representation for the non-associative product of space-time surfaces defined by the condition $RE(\prod_i P_i) = 0$. Could the scattering amplitude be constructed from products of octonion valued single particle amplitudes. This kind of condition would pose strong constraints on the theory. Could the scattering amplitudes associated with different associations be octonionic - may be differing by octonion-valued phase factors - and could only their sum be real in octonionic sense (recall that complexified octonions involving imaginary unit i commuting with the octonionic imaginary units are considered)?

One can look the situation also from the point of view of positive and negative energy states defining zero energy states as they pairs.

1. The formation of association as subset is like formation of bound state of bound states of Could each external line of zero energy state have the structure of association? Could also the internal entanglement associated with a given external line be characterized in terms of association.

Could the so called monogamy theorem stating that only two-particle entanglement can be maximal correspond to the decomposing of $n = 3$ association to one- and two-particle associations? If quantum entanglement is behind associations in cognitive sense, the cognitive meaning of association could reduce to its mathematical meaning.

An interesting question relates to the notion of identical particle: are the many-particle states of identical particles invariant under associations or do they transform by phase factor under association. Does a generalization of braid statistics make sense?

2. In ZEO based quantum measurement theory the cascade of quantum measurements proceeds from long to short scales and at each step decomposes a given system to two subsystems. The cascade stops when the reduction of entanglement is impossible: this is the case if the entanglement probabilities belong to an extension of extension of rationals characterizing the extension in question. This cascade is nothing but a formation of an association! Since only the state at the second boundary of CD changes, the natural interpretation is that state function reduction mean a selection of association in 3-D sense.
3. The division of n objects to groups has also social meaning: all social groups tend to divide into cliques spoiling the dream about full democracy. Only a group with 2 members - Romeo and Julia or Adam and Eve - can be a full democracy in practice. Already in a group of 3 members 2 members tend to form a clique leaving the third member outside. Jules and Catherine, Jim and Catherine, or maybe Jules and Jim! Only a paradise allows a full democracy in which non-associativity holds true. In ZEO it would be realized only at the quantum critical external lines of scattering diagram and quantum criticality means instability. Quantum superposition of all associations could realize this democracy in 4-D sense.

A further perspective is provided by many-sheeted space-time providing classical correlate for quantum dynamics.

1. Many-sheeted space-time means that physical states have a hierarchical structure - just like associations do. Could the formation of association (AB) correspond basically to a formation of flux tube bond between A and B to give AB and serve as space-time correlate for (ne-gentropic) entanglement. Could ((AB)C) would correspond to (AB) and (C) “topologically condensed” to a larger surface. If so, the hierarchical structure of many-sheeted space-time would represent associations and also the basic structures of language.
2. Spin glass (see <http://tinyurl.com/y9yyq8ga>) is a system characterized by so called frustrations. Spin glass as a thermodynamical system has a very large number of minima of free energy and one has fractal energy landscape with valleys inside valleys. Typically there is a competition between different pairings (associations) of the basic building bricks of the system.

Could spin glass be describable in terms of associations? The modelling of spin glass leads to the introduction of ultrametric topology characterizing the natural distance function for the free energy landscape. Interestingly, p-adic topologies are ultrametric. In TGD framework I have considered the possibility that space-time is like 4-D spin glass: this idea was originally inspired by the huge vacuum degeneracy of Kähler action. The twistor lift of TGD breaks this degeneracy but 4-D spin glass idea could still be relevant.

6 Gromov-Witten invariants, Riemann-Roch theorem, and Atiyah-Singer index theorem from TGD point of view

Gromov-Witten (G-W) invariants, Riemann-Roch theorem (RR), and Atiyah-Singer index theorem (AS) are applied in advanced algebraic geometry, and it is interesting to see whether they could have counterparts in TGD framework. The basic difference between TGD and conventional algebraic geometry is due to the adelic hierarchy demanding that the coefficients of polynomials involved are in given extension of rationals. Continuous moduli spaces are replaced with discrete ones by number theoretical quantization due to the criticality guaranteeing associativity of tangent or normal space. $M^8 - H$ duality brings in powerful consistency conditions: counting of allowed combinations of coefficients of polynomials on M^8 side and counting of dimensions on H side using AS should give same results. $M^8 - H$ duality might be in fact analogous to the mirror symmetry of M-theory.

6.1 About the analogs of Gromov-Witten invariants and branes in TGD

Gromov-Witten invariants, whose definition was discussed in [L7], play a central role in superstring theories and M-theory and are closely related to branes. For instance, partition functions can be expressed in terms of these invariants giving additional invariants of symplectic and algebraic geometries. Hence it is interesting to look whether they could be important also in TGD framework.

1. As such the definition of G-W invariants discussed in [L7] do not make sense in TGD framework. For instance, space-time surface is not a closed symplectic manifold whereas M^8 and H are analogs of symplectic spaces. Minkowskian regions of space-time surface have Hamilton-Jacobi structure at the level of both M^8 and H and this might replace the symplectic structure. Space-time surfaces are not closed manifolds.

Physical intuition however suggests that the generalization exists. The fact that Minkowskian metric and Euclidian metric for complexified octonions are obtained in various sectors for which complex valued length squared is real suggests that signature is not a problem. Kähler form for complexified z gives as special case analog of Kähler form for E^4 and M^4 .

2. The quantum intersection defines a description of interactions in terms of string world sheets. If I have understood G-W invariant correctly, one could have for $D > 4$ -dimensional symplectic spaces besides partonic $2k - 2$ -D surfaces also surfaces with smaller but even dimension identifiable as branes of various dimensions. Branes would correspond to a generalization of relative cohomology. In TGD framework one has $2k = 4$ and the partonic 2-surfaces have

dimension 2 so that classical intersections consisting of discrete points are possible and stable for string world sheets and partonic 2-surfaces. This is a unique feature of 4-D space-time.

One might think a generalization of G-W invariant allowing to see string world sheets as connecting the spaced-like 3-surfaces at the boundaries of CDs and light-like orbits of partonic 2-surfaces. The intersection is not discrete now and marked points would naturally correspond to the ends points of strings at partonic 2-surfaces associated with the boundaries of CD and with the vertices of topological scattering diagrams.

3. The idea about 2-D string world sheet as interaction region could generalize in TGD to space-time surface inside CD defining 4-D interaction region. In [L8] one indeed ends up with amazingly similar description of interactions for n external particles entering CD and represented as zero loci for quaternion valued “real” part $RE(P)$ or “imaginary” part $IM(P)$ for the complexified octonionic polynomial.

Associativity forces quantum criticality posing conditions on the coefficients of the polynomials. Polynomials with the origin of octonion coordinate along the same real axis commute and associate. Since the origins are different for external particles in the general case, the polynomials representing particles neither commute nor associate inside the interaction region defined by CD but one can also now define zero loci for both $RE(\prod P_i)$ and $IM(\prod P_i)$ giving $P_i = 0$ for some i . Now different permutations and different associations give rise to different interaction regions and amplitude must be sum over all these.

3-vertices would correspond to conditions $P_i = 0$ for 3 indices i simultaneously. The strongest condition is that 3 partonic 2-surfaces X_i^2 co-incide: this condition does not satisfy classical dimension rule and should be posed as essentially 4-D boundary condition. Two partonic 2-surfaces $X_i^2(t_i(n))$ intersect at discrete set of points: could one assume that the sparticle lines intersect and there fusion is forced by boundary condition? Or could one imagine that partonic 2-surfaces turns back in time and second partonic 2-surface intersects it at the turning point?

4. In 4-D context string world sheets are associated with magnetic flux tubes connecting partonic orbits and together with strings serve as correlates for negentropic entanglement assignable to the p-adic sectors of the adèle considered, to attention in consciousness theory, and to remote mental interactions in general and occurring routinely between magnetic body and biological body also in ordinary biology. This raises the question whether “quantum touch” generalizes from 2-D string world sheets to 4-D space-time surface (magnetic flux tubes) connecting 3-surfaces at the orbits and partonic orbits.
5. The above formulation applies to closed symplectic manifolds X . One can however generalize the formulation to algebraic geometry. Now the algebraic curve X^2 is characterized by genus g and order of polynomial n defining it. This formulation looks very natural in M^8 picture.

An interesting question is whether the notion of brane makes sense in TGD framework.

1. In TGD branes inside space-time variety are replaced by partonic 2-surfaces and possibly by their light-like orbits at which the induced metric changes signature. These surfaces are metrically 2-D. String world sheets inside space-time surfaces have discrete intersection with the partonic 2-surfaces. The intersection of strings as space-like *resp.* light-like boundaries of string world sheet with partonic orbit sheet *resp.* space-like 3-D ends of space-time surface at boundaries of CD is also discrete classically.
2. An interesting question concerns the role of 6-spheres $S^6(t_n)$ appearing as special solutions to the octonionic zero locus conditions solving both $RE(P_n) = 0$ and $IM(P_n) = 0$ requiring $P_n(o) = 0$. This can be true at 7-D light cone $o = et$, e light-like vector and t a real parameter. The roots t_n of $P(t) = 0$ give 6-spheres $S^6(t_n)$ with radius t_n as solutions to the singularity condition. As found, one can assign to each factor P_i in the product of polynomials defining many-particle state in interaction region its own partonic 2-surfaces $X^2(t_n)$ related to the solution of $P_i(t) = 0$

Could one interpret 6-spheres as brane like objects, which can be connected by 2-D “free” string world sheets as 2-varieties in M^8 and having discrete intersection with them implied

by the classical dimension condition for the intersection. Free string world sheets would be something new and could be seen as trivially associative surfaces whereas 6-spheres would represent trivially co-associative surfaces in M^8 .

The 2-D intersections of $S^6(t_n)$ with space-time surfaces define partonic 2-surfaces X^2 appearing at then ends of space-time and as vertices of topological diagrams. Light-like sparticle lines along parton orbits would fuse at the partonic 2-surfaces and give rise to the analog of 3-vertex in $\mathcal{N} = 4$ SUSY.

Some further TGD inspired remarks are in order.

1. Virasoro conjecture generalizing Witten conjecture involves half Virasoro algebra. Super-Virasoro algebra algebra and its super-symplectic counterpart (SSA) play a key role in the formulation of TGD at level of H . Also these algebras are half algebras. The analogs of super-conformal conformal gauge conditions state that sub-algebra of SSA with conformal weights coming as n -ples of those for entire algebra and its commutator with entire SSA give rise to vanishing Noether charges and annihilate physical states.

These conditions are conjecture to fix the preferred extremals and serve as boundary conditions allowing the formulation of $M^8 - H$ correspondence inside space-time regions (interaction regions), where the associativity conditions fail to be true and direct $M^8 - H$ correspondence does not make sense. Non-trivial solutions to these conditions are possible only if one assumes half super-conformal and half super-symplectic algebras. Otherwise the generators of the entire SSA annihilate the physical states and all SSA Noether charges vanish. The invariance of partition function for string world sheets in this sense could be interpreted in terms of emergent dynamical symmetries.

2. Just for fun one can consider the conjecture that the reduction of quantum intersections to classical intersections mediated by string world sheets implies that the numbers of string world sheets as given by the analog of G-W invariants are integers.

6.2 Does Riemann-Roch theorem have applications to TGD?

Riemann-Roch theorem (RR) (see <http://tinyurl.com/mdmbcx6>) is a central piece of algebraic geometry. Atiyah-Singer index theorem is one of its generalizations relating the solution spectrum of partial differential equations and topological data. For instance, characteristic classes classifying bundles associated with Yang-Mills theories (see <http://tinyurl.com/y9xvkhyy>) have applications in gauge theories and string models.

The advent of octonionic approach to the dynamics of space-time surfaces inspired by $M^8 - H$ duality [L6] [L7, L8] gives hopes that dynamics at the level of complexified octonionic M^8 could reduce to algebraic equations plus criticality conditions guaranteeing associativity for space-time surfaces representing external particles, in interaction region commutativity and associativity would be broken. The complexification of octonionic M^8 replacing norm in flat space metric with its complexification would unify various signatures for flat space metric and allow to overcome the problems due to Minkowskian signature. Wick rotation would not be a mere calculational trick.

For these reasons time might be ripe for applications of possibly existing generalization of RR to TGD framework. In the following I summarize my admittedly unprofessional understanding of RR discussing the generalization of RR for complex algebraic surfaces having real dimension 4: this is obviously interesting from TGD point of view.

I will also consider the possible interpretation of RR in TGD framework. One interesting idea is possible identification of light-like 3-surfaces and curves (string boundaries) as generalized poles and zeros with topological (but not metric) dimension one unit higher than in Euclidian signature.

6.2.1 Could a generalization of Riemann-Roch theorem be useful in TGD framework?

The generalization of RR for algebraic varieties, in particular for complex surfaces (real dimension equal to 4) exists. In M^8 picture the complexified metric Minkowskian signature need not cause any problems since the situation can be reduced to Euclidian sector. Clearly, this picture would provide a realization of Wick rotation as more than a trick to calculate scattering amplitudes.

Consider first the motivations for the desire of having analog of Riemann-Roch theorem (RR) at the level of space-time surfaces in M^8 .

1. It would be very nice if partonic 2-surfaces would have interpretation as analogs of zeros or poles of a meromorphic function. RR applies to the divisors characterizing meromorphic functions and 2-forms, and one could hope of obtaining information about the dimensions of these function spaces giving rise to octonionic space-time varieties. Note however that the reduction to real polynomials or even rational functions might be already enough to give the needed information. Rational functions are required by the simplest generalization whereas the earlier approach assumed only polynomials. This generalization does not however change the construction of space-time varieties as zero loci of polynomials in an essential manner as will be found.
2. One would like to count the degeneracies for the intersections of 2-surfaces of space-time surface and here RR might help since its generalization to complex surfaces involves intersection form as was found in the brief summary of RR for complex surfaces with real dimension 4 (see Eq. ??).

In particular, one would like to know about the intersections of partonic 2-surfaces and string world sheets defining the points at which fermions reside. The intersection form reduces the problem via Poincare duality to 2-cohomology of space-time surfaces. More generally, it is known that the intersection form for 2-surfaces tells a lot about the topology of 4-D manifolds (see <http://tinyurl.com/y8tmqtef>). This conforms with SH. Gromow-Witten invariants [L2] (see <http://tinyurl.com/ybobccub>) are more advanced rational valued invariants but might reduce to integer valued invariants in TGD framework [L8].

There are also other challenges to which RR might relate.

1. One would like to know whether the intersection points for string world sheets and partonic 2-surfaces can belong in an extension of rationals used for adèle. If the points belong to cognitive representations and subgroup of Galois group acts trivially then the number of points is reduced as the points at its orbit fuse together. The sheets of the Galois covering would intersect at point. The images of the fused points in H could be disjoint points since tangent spaces need not be parallel.
2. One would also like to have idea about what makes partonic 2-surfaces and string world sheets so special. In 2-D space-time one would have points instead of 2-surfaces. The obvious idea is that at the level of M^8 these 2-surfaces are in some sense analogous to poles and zeros of meromorphic functions. At the level of H the non-local character of $M^8 - H$ would imply that preferred extremals are solutions of an action principle giving partial differential equations.

6.2.2 What could be the analogs of zeros and poles of meromorphic function?

The basic challenge is to define what notions like pole, zero, meromorphic function, and divisor could mean in TGD context. The most natural approach based on a simple observation that rational functions need not define map of space-time surface to itself. Even though rational function can have pole inside CD, the point ∞ need not belong to the space-time variety defined the rational functions. Hence one can try the modification of the original hypothesis by replacing the octonionic polynomials with rational functions. One cannot exclude the possibility that although the interior of CD contains only finite points, the external particles outside CD could extend to infinity.

1. For octonionic analytic polynomials the notion of zero is well-defined. The notion of pole is well-defined only if one allows rational functions $R = P_1(o)/P_2(o)$ so that poles would correspond to zeros for the denominator of rational function. 0 and ∞ are both unaffected by multiplication and ∞ also by addition so that they are algebraically special. There are several variants of this picture. The most general option is that for a given variety zeros of both P_i are allowed.
2. The zeros of $IM(P_1) = 0$ and $IM(P_2) = 0$ would give solutions as unions of surfaces associated with P_i . This is because $IM(o_1 o_2) = IM(o_1)RE(o_2) + IM(o_2)RE(o_1)$. There is no

need to emphasize how important this property of IM for product is. One might say that one has two surfaces which behave like free non-interacting particles.

3. These surfaces should however interact somehow. The intuitive expectation is that the two solutions are glued by wormhole contacts connecting partonic 2-surfaces corresponding to $IM(P_1) = 0$ and $IM(P_2) = 0 = \infty$. For $RE(P_i) = 0$ and $RE(P_i) = \infty$ the solutions do not reduce to separate solutions $RE(P_1) = 0$ and $RE(P_2) = 0$. The reason is that the real part of $o_1 o_2$ satisfies $Re(o_1 o_2) = Re(o_1)Re(o_2) - Im(o_1)Im(o_2)$. There is a genuine interaction, which should generate the wormhole contact. Only at points for which $P_1 = 0$ and $P_2 = 0$ holds true, $RE(P_1) = 0$ and $RE(P_2) = 0$ are satisfied simultaneously. This happens in the discrete intersection of partonic 2-surfaces.
4. Elementary particles correspond even for $h_{eff} = h$ to two-sheeted structures with partonic surfaces defining wormhole throats. The model for elementary particles requires that particles are minimally 2-sheeted structures since otherwise the conservation of monopole Kähler magnetic flux cannot be satisfied: the flux is transferred between space-time sheets through wormhole contacts with Euclidian signature of induced metric and one obtains closed flux loop. Euclidian wormhole contact would connect the two Minkowskian sheets. Could the Minkowskian sheets correspond to zeros $IM(P_i)$ for P_1 and P_2 and could wormhole contacts emerge as zeros of $RE(P_1/P_2)$?

One can however wonder whether this picture could allow more detailed specification. The simplest possibility would be following. The basic condition is that CD emerges automatically from this picture.

1. The simplest possibility is that one has $P_1(o)$ and $P_2(T - o)$ with the origin of octonions at the “lower” tip of CD. One would have $P_1(0) = 0$ and $P_2(0) = 0$. $P_1(o)$ would give rise to the “lower” boundary of CD and $P_2(T - o)$ to the “upper” boundary of CD.

ZEO combined with the ideas inspired by infinite rationals as counterparts of space-time surfaces connecting 3-surfaces at opposite boundaries of CD [K19] would suggest that the opposite boundaries of CD could correspond zeros and poles respectively and the ratio $P_1(o)/P_2(T - o)$ and to zeros of P_1 resp. P_2 assignable to different boundaries of CD. Both light-like parton orbits and string world sheets would interpolate between the two boundaries of CD at which partonic 2-surface would correspond to zeros and poles.

The notion divisor would be a straightforward generalization of this notion in the case of complex plane. What would matter would be the rational function $P_1(t)/P_2(T - t)$ extended from the real (time) axis of octonions to the entire space of complexified octonions. Positive degree of divisor would multiply $P_1(t)$ with $(t - t_1)^m$ inducing a new zero at or increasing the order of existing zero at t_1 . Negative orders n would multiply the denominator by $(t - t_1)^n$.

2. One can also consider the possibility that both boundaries of CD emerge for both P_1 and P_2 and without assigning either boundary of CD with P_i . In this case P_i would be sum over terms $P_{ik} = P_{ia_k}(o)P_{ib_k}(T - o)$ of this kind of products satisfying $P_{ia_k}(0) = 0$ and $P_{ib_k}(0) = 0$.

One can imagine also an alternative approach in which 0 and ∞ correspond to opposite tips of CD and have geometric meaning. Now zeros and poles would correspond to 2-surfaces, which need not be partonic. Note that in the case of Riemann surfaces ∞ can represent any point. This approach does not however look attractive.

6.2.3 Could one generalize RR to octonionic algebraic varieties?

RR is associated with complex structure, which in TGD framework seems to make sense independent of signature thanks to complexification of octonions. Divisors are the key notion and characterize what might be called local winding numbers. De-Rham cohomology is replaced with much richer Dolbeault cohomology (see <http://tinyurl.com/y7cvs5sx>) since the notion of continuity is replaced with that of meromorphy. Symplectic approach about which G-W invariants for symplectic manifolds provide an example define a different approach and now one has ordinary cohomology.

An interesting question is whether $M^8 - H$ -duality corresponds to the mirror symmetry of string models (see <http://tinyurl.com/yc2m2e5m>) relating complex structures and symplectic structures. If this were the case, M^8 would correspond to complex structure and H to symplectic structure.

RR for curves gives information about dimensions for the spaces of meromorphic functions having poles with order not higher than specified by divisor. This kind of interpretation would be very attractive now since the poles and zeros represented as partonic 2-surfaces would have direct physical interpretation in terms of external particles and interaction vertices. RR for curves involves poles with orders not higher than specified by the divisor and gives a formula for the dimension of the space of meromorphic functions for a given divisor. As a special case give the dimension $l(nD)$ for a given divisor.

Could something similar be true in TGD framework?

1. Arithmetic genus makes sense for polynomials $P(t)$ since t can be naturally complexified giving a complex curve with well-defined arithmetic genus. What could correspond to the intersection form for 2-surfaces representing D and $K - D$? The most straightforward possibility is that partonic 2-surfaces correspond to poles and zeros.

Divisor $-D$ would correspond to the inverse of P_2/P_1 representing it. $D - K$ would also have a well-defined meaning provided the canonical divisor associated with holomorphic 2-form has well-defined meaning in the Dolbeault cohomology of the space-time surface with complex structure. RR would give direct information about the space of space-time varieties defined by $RE(P) = 0$ or $IM(P) = 0$ condition.

One could hope of obtaining information about intersection form for string world sheets and partonic 2-surfaces. Whether the divisor $D - K$ has anything to do with string world sheets, is of course far from clear.

2. Complexification means that field property fails in the sense that complexified Euclidian norm vanishes and the inverse of complexified octonion/quaternion/complex number is infinite formally. For Euclidian sector with real coordinates this does not happen but does take place when some coordinates are real and some imaginary so that signature is effectively Minkowskian signature.

At 7-D light-cone of M^8 the condition $P(o) = 0$ reduces to a condition for real polynomial $P(t) = 0$ giving roots t_n . Partonic 2-varieties are intersections of 4-D space-time varieties with 6-spheres with radii t_n . There are good reasons to expect that the 3-D light-like orbits of partonic 3-surfaces are intersections of space-time variety with 7-D light-cone boundary and their H counterparts are obtained as images under $M^8 - H$ duality.

For light-like complexified octonionic points the inverse of octonion does not exist since the complexified norm vanishes. Could the light-like 3-surfaces as partonic orbits correspond to images under $M^8 - H$ duality for zeros and/or poles as 3-D light-like surfaces? Could also the light-like boundaries of strings correspond to this kind of generalized poles or zeros? This could give a dynamical realization for the notions of zero and pole and increase the topological dimension of pole and zero for both 2-varieties and 4-varieties by one unit. The metric dimension would be unaffected and this implies huge extension of conformal symmetries central in TGD since the light-like coordinate appears as additional parameter in the infinitesimal generators of symmetries.

Could one formulate the counterpart of RR at the level of H ? The interpretation of $M^8 - H$ duality as analog of mirror symmetry (see <http://tinyurl.com/yc2m2e5m>) suggests this. In this case the first guess for the identification of the counterpart of canonical divisor could be as Kähler form of CP_2 . This description would provide symplectic dual for the description based on divisors at the level of M^8 . G-W invariants and their possible generalization are natural candidates in this respect.

6.3 Could the TGD variant of Atiyah-Singer index theorem be useful in TGD?

Atiyah-Singer index theorem (AS) is one of the generalizations of RR and has shown its power in gauge field theories and string models as a method to deduce the dimensions of various moduli spaces for the solutions of field equations. A natural question is whether AS could be useful in TGD and whether the predictions of AS at H side could be consistent with $M^8 - H$ duality suggesting very simple counting for the numbers of solutions at M^8 side as coefficient combinations of polynomials in given extension of rationals satisfying criticality conditions. One can also ask whether the hierarchy of degrees n for octonion polynomials could correspond to the fractal hierarchy of generalized conformal sub-algebras with conformal weights coming as n -multiples for those for the entire algebras.

Atiyah-Singer index theorem (AS) and other generalizations of RR involve extremely abstract concepts. The best manner to get some idea about AS is to learn the motivations for it. The article <http://tinyurl.com/yc4911jp> gives a very nice general view about the motivations of Atiyah-Singer index theorem and also avoids killing the reader with details.

Solving problems of algebraic geometry is very demanding. The spectrum of solutions can be discrete (say number of points of space-time surface having linear M^8 coordinates in an extension of rationals) or continuous such as the space of roots for n :th order polynomials with real coefficients.

An even more difficult challenge is solving of partial differential equations in some space, call it X , of say Yang-Mills gauge field coupled to matter fields. In this case the set of solutions is typically continuous moduli space.

One can however pose easier questions. What is the number of solutions in counting problem? What is the dimension of the moduli space of solutions? Atiyah-Singer index theorem relates this number - analytic index - to topological index expressible in terms of topological invariants assignable to complexified tangent bundle of X and to the bundle structure - call it field bundle - accompanying the fields for which field equations are formulated.

6.3.1 AS very briefly

Consider first the assumptions of AS.

1. The idea is to study perturbations of a given solution and linearize the equations in some manifold X often assumed to be compact. This leads to a linear partial differential equations defined by linear operator P . One can deduce the dimension of the solution space of P . This number defines the dimension of the tangent space of solution space of full partial differential equations, call it moduli space.
2. The idea is to assign to the partial differential operator P its symbol $\sigma(P)$ obtained by replacing derivatives with what might be called momentum components. The reversal of this operation is familiar from elementary wave mechanics: $p_i \rightarrow id/dx^i$. This operation can be formulated in terms of co-tangent bundle. The resulting object is purely algebraic. If this matrix is reversible for all momentum values and points of X , one says that the operator is elliptic.

Note that for field equations in Minkowski space M^4 the invertibility constraint is not satisfied and this produces problems. For instance, for massive M^4 d'Alembertian for scalar field the symbol is four-momentum squared, which vanishes, when on-mass shell condition is satisfied. Wick rotation is somewhat questionable manner to escape this problem. One replaces Minkowski space with its Euclidian counterpart or by 4-sphere. If all goes well the dimension of the solution space does not depend on the signature of the metric.

3. In the general case one studies linear equation of form $DP = f$, where f is homogeneity term representing external perturbation. f can also vanish. Quite generally, one can write the dimension of the solution space as

$$Ind_{anal}(P) = dim(ker(P)) - dim(coker(P)) \quad (6.1)$$

$ker(P)$ denotes the solution space for $DP = 0$ without taking into account the possible restrictions coming from the fact that f can involve part f_0 satisfying $Df_0 = 0$ (for instance, f_0 corresponds to resonance frequency of oscillator system) nor boundary conditions guaranteeing hermiticity. Indeed, the hermitian conjugate D^\dagger of D is not automatically identical with D . D^\dagger is defined in terms of the inner product for small perturbations as

$$\langle D^\dagger P_1^* | DP_2 \rangle = \langle P_1 | DP_2 \rangle . \tag{6.2}$$

The inner product involves integration over X and partial integrations transfer the action of partial derivatives from P_2 to P_1^* . This however gives boundary terms given by surface integral and hermiticity requires that they vanish. This poses additional conditions on P and contributes to $dim(coker(P))$.

The challenge is to calculate $Ind_{anal}(P)$ and here AS is of enormous help. AS relates analytical index $Ind_{anal}(P)$ for P to topological index $Ind_{top}(\sigma(P))$ for its symbol $\sigma(P)$.

1. $Ind_{top}(\sigma(P))$ involves only data associated with the topology X and with the bundles associated with field variables. In the case of Yang-Mills fields coupled to matter the bundle is the bundle associated with the matter fields with a connection determined by Yang-Mills gauge potentials. So called Todd class $Td(X)$ brings in information about the topology of complexified tangent bundle.
2. $Ind_{top}(\sigma(P))$ is not at all easy to define but is rather easily calculable as integrals of various invariants assignable to the bundle structure involved. Say instanton density for YM fields and various topological invariants expressing the topological invariants associated with the metric of the space. What is so nice and so non-trivial is that the dimension of the moduli space for non-linear partial differential equations is determined by topological invariants. Much of the dynamics reduces to topology.

The expression for $Ind_{top}(\sigma(P))$ involves besides σ_P topological data related to the field bundle and to the complexified tangent bundle. The expression Ind_{top} as a function of the symbol $\sigma(P)$ is given by

$$Ind_{top}(\sigma(P)) = (-1)^n \langle ch(\sigma(P)) \cdot Td(T_C(X), [X]) \rangle . \tag{6.3}$$

The expression involves various topological data.

1. Dimension of X .
2. The quantity $\langle x.y \rangle$ involving cup product $x.y$ of cohomology classes, which contains a contribution in the highest homology group $H^n(X)$ of X corresponding to the dimension of X and is contracted with this fundamental class $[X]$. $\langle x.y \rangle$ denotes matrix trace for the operator $ch(\sigma(P))$ formed as polynomial of $\sigma(P)$. $[X]$ denotes so called fundamental class for X belonging to H^n and defines the orientation of X .
3. Chern character $ch_E(t)$ (see <http://tinyurl.com/ybavu66h>). I must admit that I ended up to a garden of branching paths while trying to understand the definition of ch_E is. In any case, $ch_E(t)$ characterizes complex vector bundle E expressible in terms of Chern classes (see <http://tinyurl.com/y8j1aznc>) of E . E is the bundle assignable to field variables, say Yang Mills fields and various matter fields.

Both direct sums and tensor products of fiber spaces of bundles are possible and the nice feature of Chern class is that it is additive under tensor product and multiplicative under direct sum. The fiber space of the entire bundle is now direct sum of the tangent space of X and field space, which suggests that $Ind(top)$ is actually the analog of Chern character for the entire bundle.

$t = \sigma P$ has interpretation as an argument appearing in the definition of Chern class generalized to Chern character. $t = \sigma(P)$ would naturally correspond to a matrix valued argument

of the polynomial defining Chern class as cohomology element. $ch(\sigma(P))$ is a polynomial of the linear operator defined by symbol $\sigma(P)$. ch_E for given complex vector bundle is a polynomial, whose coefficients are relatively easily calculable as topological invariants assignable to bundle E . E must be the field bundle now.

4. Todd class $Td(T_C(X))$ for the complexified tangent bundle (see <http://tinyurl.com/yckv4w84>) appears also in the expression. Note that also now the complexification occurs. The cup product gives element in $H^n(X)$, which is contracted with fundamental class $[X]$ and integrated over X .

6.3.2 AS and TGD

The dynamics of TGD involves two levels: the level of complexified M^8 (or equivalently E^8) and the level of H related to $M^8 - H$ correspondence.

1. At the level of M^8 one has algebraic equations rather than partial differential equations and the situation is extremely simple as compared to the situation for a general action principle. At the level of H one has action principle and partial differential equations plus infinite number of gauge conditions selecting preferred extremals and making dynamics for partial differential equations dual to the dynamics determined by purely number theoretic conditions.

The space-time varieties representing external particles outside CDs in M^8 satisfy associativity conditions for tangent space or normal space and reducing to criticality conditions for the real coefficients of the polynomials defining the space-time variety. In the interior of CDs associativity conditions are not satisfied but the boundary conditions fix the values of the coefficients to be those determined by criticality conditions guaranteeing associativity outside the CD.

In the interiors space-time surfaces of CDs M^8 -duality does not apply but associativity of tangent spaces or normal spaces at the boundary of CD fixes boundary values and minimal surface dynamics and strong form of holography (SH) fixes the space-time surfaces in the interior of CD.

2. For the H -images of space-time varieties in H under $M^8 - H$ duality the dynamics is universal coupling constant independent critical dynamics of minimal surfaces reducing to holomorphy in appropriate sense. For minimal surfaces the 4-D Kähler current density vanishes so that the solutions are 4-D analogs of geodesic lines outside CD. Inside CD interactions are coupled on and this current is non-vanishing. Infinite number of gauge conditions for various half conformal algebras in generalized sense code at H side for the number theoretical critical conditions at M^8 side. The sub-algebra with conformal weights coming as n -ples of the entire algebra and its commutator with entire algebra gives rise to vanishing classical Noether charges. An attractive assumption is that the value of n at H side corresponds to the order n of the polynomials at M^8 side.
3. The coefficients of polynomials $P(o)$ determining space-time varieties are real numbers (also complexified reals can be considered without losing associativity) restricted to be numbers in extension of rationals. This makes it possible to speak about p-adic variants of the space-time surfaces at the level of M^8 at least.

Could Atiyah-Singer theorem have relevance for TGD?

1. For real polynomials it is easy to calculate the dimension of the moduli space by counting the number of independent real (in octonionic sense) coefficients of the polynomials of real variable (one cannot exclude that the coefficients are in complex extension of rationals). Criticality conditions reduce this number and the condition that coefficients are in extension of rationals reduces it further. One has quite nice overall view about the number of solutions and one can see them as subset of continuous moduli space. If $M^8 - H$ duality really works then this gives also the number of preferred extremals at H side.

2. This picture is not quite complete. It assumes fixing of 8-D CD in M^8 as well as fixing of the decomposition $M^2 \subset M^4 \subset M^4 \times E^4$. This brings in moduli space for different choices of octonion structures (8-D Lorentz group is involved). Also moduli spaces for partonic 2-surfaces are involved. Number theoretical universality seems to require that also these moduli spaces have only points with coordinates in extension of rationals involved.
3. In principle one can try to formulate the counterpart of AS at H side for the linearization of minimal surface equations, which are nothing but the counterpart of massless field equations in a fixed background metric. Note that additional conditions come from the requirement that the term from Kähler action reduces to minimal surface term.

Discrete sets of solutions for the extensions of rationals should correspond to each other at the two sides. One can also ask whether the dimensions for the effective continuous moduli spaces labelled by n characterizing the sub-algebras of various conformal algebras isomorphic to the entire algebra and those for the polynomials of order n satisfying criticality conditions. One would have a number theoretic analog for a particle in box leading to the quantization of momenta.

All this is of course very speculative and motivated only by the general physical vision. If the speculations were true, they would mean huge amount of new mathematics.

7 Intersection form for 4-manifolds, knots and 2-knots, smooth exotics, and TGD

Gary Ehlenberger sent a highly interesting commentary related to smooth structures in R^4 discussed in the article of Gompf [A7] (<https://cutt.ly/eMracmf>) and more generally to exotics smoothness discussed from the point of view of mathematical physics in the book of Asselman-Maluga and Brans [A8] (<https://cutt.ly/DMu0dYr>). I am grateful for these links for Gary.

7.1 Basic ideas

7.1.1 The role of intersection forms in TGD

The intersection form of 4-manifold (<https://cutt.ly/jMriNdI>) characterizing partially its 2-homology is a central notion in the study of the smooth structures. I am not a topologist but have two good reasons to get interested on intersection forms.

1. In the TGD framework [L17], the intersection form describes the intersections of string world sheets and partonic 2-surfaces and therefore is of direct physical interest [K6, L8].
2. Knots have an important role in TGD. The 1-homology of the knot complement characterizes the knot. Time evolution defines a knot cobordism as a 2-surface consisting of knotted string world sheets and partonic 2-surfaces. A natural guess is that the 2-homology for the 4-D complement of this cobordism characterizes the knot cobordism. Also 2-knots are possible in 4-D space-time and a natural guess is that knot cobordism defines a 2-knot.

The intersection form for the complement for cobordism as a way to classify these two-knots is therefore highly interesting in the TGD framework. One can also ask what the counterpart for the opening of a 1-knot by repeatedly modifying the knot diagram could mean in the case of 2-knots and what its physical meaning could be in the TGD Universe. Could this opening or more general knot-cobordism of 2-knot take place in zero energy ontology (ZEO) [L13, L16, L18] as a sequence of discrete quantum jumps leading from the initial 2-knot to the final one.

7.1.2 Why exotic smooth structures are not possible in TGD?

The existence of exotic 4-manifolds [A7, A8, A2] could be an anomaly in the TGD framework. In the articles [A7, A2] the term anomaly is indeed used. Could these anomalies cancel in the TGD framework?

The first naive guess was that the exotic smooth structures are not possible in TGD but it turned out that this is not trivially true. The reason is that the smooth structure of the space-time surface is not induced from that of H unlike topology. One could induce smooth structure by assuming it given for the space-time surface so that exotics would be possible. This would however bring an ad hoc element to TGD. This raises the question of how it is induced.

1. This led to the idea of a holography of smoothness, which means that the smooth structure at the boundary of the manifold determines the smooth structure in the interior. Suppose that the holography of smoothness holds true. In ZEO, space-time surfaces indeed have 3-D ends with a unique smooth structure at the light-like boundaries of the causal diamond $CD = cd \times CP_2 \subset H = M^4 \times CP_2$, where cd is defined in terms of the intersection of future and past directed light-cones of M^4 . One could say that the absence of exotics implies that $D = 4$ is the maximal dimension of space-time.
2. The differentiable structure for $X^4 \subset M^8$, obtained by the smooth holography, could be induced to $X^4 \subset H$ by $M^8 - H$ -duality. Second possibility is based on the map of mass shell hyperboloids to light-cone proper time $a = \text{constant}$ hyperboloids of H belonging to the space-time surfaces and to a holography applied to these.
3. There is however an objection against holography of smoothness (<https://cutt.ly/3MewY0t>). In the last section of the article, I develop a counter argument against the objection. It states that the exotic smooth structures reduce to the ordinary one in a complement of a set consisting of arbitrarily small balls so that local defects are the condensed matter analogy for an exotic smooth structure.

7.2 Intersection form in the case of 4-surfaces

Intersection form (<https://cutt.ly/jMrINdI>) for homologically trivial 2-surfaces of the space-time surface and 2-homology for the complement of these surfaces can be physically important in tGD framework.

7.2.1 Intersection forms in 2-D case

It is good to explain the notion of intersection form by starting from 1-homology. The intersection form for 1-homology is encountered for a cylinder with ends fixed. In this case, one has relative homology and homologically trivial curves are curves connecting the ends of string and characterized by a winding number.

In the case of torus obtained by identifying the ends of cylinder, one obtains two winding numbers (m, n) corresponding to homologically non-trivial circles at torus. The intersection number for curves (m, n) and (p, q) at torus is $N = mq - np$ and for curves at cylinder one as $(m, n) = (1, n)$ giving $N = n - q$.

The antisymmetric intersection form is defined as 2×2 matrix defining intersections for the basis of the homology with $(m, n) = (1, 0)$ and $(n, m) = (0, 1)$ and is given by $(0, 1; -1, 0)$.

7.2.2 Intersection for 4-surfaces in TGD context

In TGD, the intersection form for a 4-surface identified as space-time surface could have a rather concrete physical interpretation and the stringy part of TGD physics would actually realize it concretely.

1. $M^8 - H$ duality requires that the 4-surface in M^8 has quaternionic/associative normal space: this distribution of normal spaces is integrable and integrates to the 4-surface in M^8 .

The normal must also contain a commutative (complex) sub-space at each point. Only this allows us to parametrize normal spaces by points of CP_2 and map them to space-time surfaces in $H = M^4 \times CP_2$. The integral distribution of these commutative sub-spaces defines a 2-D surface. Physically, these surfaces would correspond to string world sheets and partonic 2-surfaces.

- String world sheets and partonic 2-surfaces, regarded as objects in relative homology (modulo ends of the space-time surfaces at the boundaries of causal diamond (CD)), can intersect as 2-D objects inside the space-time surface and the intersection form characterizes them.

There is an analogy with the cylinder: time-like direction corresponds to the cylinder axis and a homologically non-trivial 2-surface of CP_2 corresponds to the circle at the cylinder.

- If the second homology of the space-time surface is trivial, the naive expectation is that the intersections of string world sheets are not stable under large enough deformations of the string world sheets. Same applies to intersecting plane curves. At the cylinder, the situation is different since the relative first homology is non-trivial and spanned by two generators: the circle and a line connecting the ends of the cylinder.

The intersection form is however non-trivial as in the case of the cylinder for 2-surfaces having 2-D homologically non-trivial CP_2 projection. They would represent M^4 deformations of 2-D homologically trivial surfaces of CP_2 just like a helical orbit along a cylinder surface. A 2-D generalization of CP_2 type extremal would have a light-like curve or light-like geodesic as M^4 projection and could define light-partonic orbit.

- The intersection of string world sheet and partonic 2-surface can be stable however. Partonic 2-surface is a boundary of a wormhole contact connecting two space-time sheets.

Consider a string arriving along space-time sheet A, going through the wormhole contact, and continuing along sheet B. The string has an intersection point with both wormhole throats. This intersection is stable against deformations. The orbit of this string intersects the light-like orbit of the partonic 2-surface along the light-like curve.

One has a non-trivial intersection form with the number of intersections with partonic 2-surfaces equal to 1. In analogy with cylinder, also the intersections of 2-surfaces with 2-D homologically trivial CP_2 projection are unavoidable and reflect the non-trivial intersection form of CP_2 .

7.3 About ordinary knots

Ordinary knots and 3-topologies are related and the natural expectation is that also 2-knots and 4-topologies are related.

7.3.1 About knot invariants

Consider first knot invariants (<https://cutt.ly/DMrgs14>) at the general level.

- One important knot invariant of ordinary knots is the 1-homology of the complement and the associated first homotopy group whose abelianization gives the homology group.
- The complement of the knot can be given a metric of a hyperbolic 3-manifold, which corresponds to a unit cell for a tessellation of the mass shell. $M^8 - H$ duality suggests that the intersection X^3 of 4-surface of M^8 with mass shell $H_m^3 \subset M^4 \subset M^8$ is a hyperbolic manifold and identical with the hyperbolic manifold associated with the complement of a knot of H_a^3 realized as light-cone proper time $a = \text{constant}$ hyperboloid of $M^4 \subset H$ and closed knotted and linked strings as ends of string world sheets at H_a^3 .

The evolution of the strings defined by the string world sheets would define a 1-knot cobordism. The 2-homology of the knot complement should characterize the topological evolution of the 1-homology of the knot.

7.3.2 Opening of knots and links by knot cobordisms

The procedure leading to the trivialization of knot or link can be used to define knot invariants and the procedure itself characterizes knot.

- Ordinary knot is described by a knot diagram obtained as a projection of the knot to the plane. It contains intersections of lines and the intersection contains information telling which line is above and which line is below.

2. The opening of the knot or link to give a trivial knot or link, which is used in the construction of knot invariants, is a sequence of violent operations. In the basic step strings portions go through each other and therefore suffer a reconnection. This operation can therefore change the 1-homology of the 3-D knot complement.

Knot or link can be modified by forcing two intersecting strands of the plane projection to go through each other. Locally the basic operation for two links is the same as for the pieces of knot. The transformation of the knot or link to a trivial knot or link corresponds to some sequence of these operations and can be used to define a knot invariants. This operation is not unique since there are moves which do not affect the knot.

The basic opening operation can be also seen as a time evolution, knot cobordism, in which the first portion, call it A , remains unchanged and the second portion, call it B , draws a 2-D surface in E^3 . A intersects the 2-D orbit at a single point.

3. The 2-homology for the string world sheets and partonic 2-surfaces as 2-surfaces in space-time serves as an invariant of knot cobordism and represents the topological dynamics of ordinary 1-knots of 3-surface and links formed by strings or flux tubes in 3-surface as cobordism defining the time evolution of a knot to another knot.

In particular, the intersection form for the 2-homology of the complement of the cobordism defines an invariant of cobordism. This intersection form must be distinguished from the intersection form for the second homology of the space-time surface rather than the 2-knot complement.

4. One can also consider more general sequences of basic operations transforming two knots or links to each other as knot-/link cobordisms, which involve self intersections of the knots. Does this mean that the intersection form characterizes the knot cobordism. Could a string diagram involving reconnections describe the cobordism process.

7.3.3 Stringy description of knot cobordisms

$M^8 - H$ duality [L14, L15, L20, L19] requires string world sheets and partonic 2-surfaces. This implies that TGD physics represents the 2-homology of both space-time surfaces and the homology of the complement of the knotted links defined by them.

Although the "non-homological" intersections of string world sheets can be eliminated by a suitable deformation of the string world sheet, they should have a physical meaning. This comes from the observation that they affect nontrivially the 1-homology of the knot complement as 3-D time=constant slice.

The first thing that I am able to imagine is that strings reconnect. This is nothing but the trouser vertex for strings so that intersection form would define topological string dynamics in some sense. These reconnections play a key role in TGD, also in TGD inspired quantum biology.

The dynamics of partonic 2-surfaces and string world sheets could relate to knot cobordisms, possibly leading to the opening of ordinary knot,

7.4 What about 2-knots and their cobordisms?

2-D closed surfaces in 4-D space give rise to 2-knots. What is the physical meaning of 2-knots of string world sheets? What could 2-knots for orbits of linear molecules or associated magnetic flux tubes mean physically and from the point of view of quantum information theory? One can try to understand 2-knots by generalizing the ideas related to the ordinary knots.

1. Intuitively it seems that the cobordism of a 1-knot defines a 2-knot. It is not clear to me whether all 2-knots for space-time surfaces connecting the boundaries of CD can be regarded as this kind of cobordisms of 1-knots.
2. The 2-homology of the complement of 2-knot should define a 2-knot invariant. In particular, the intersection form should define a 2-knot invariant.

3. The opening of 1-knot by repeating the above described basic operation is central in the construction of knot invariants and the sequence of the operations can be said to be knot invariant modulo moves leaving the knot unaffected.

The opening or a more general cobordism of a 2-knot could be seen as a time evolution with respect to a time parameter t_5 parametrizing the isotopy of space-time surface. The local cobordism can keep the first portion of 2-knot, call it A , unchanged and deform another portion, call it B , so that a 3-D orbit at the space-time surface is obtained. For each value of t_5 , the portions A and B of 2-knot have in the generic case only points as intersections.

This would suggest that an intersection point of A and B is generated in the operation and moves during the t_5 time evolution along A along 1-D curve during the process. This process would be the basic operation used repeatedly to open 2-knot or to transform it to another 2-knot.

4. In quantum TGD, a sequence of quantum jumps, quantum cobordism, would have the same effect as t_5 time evolution. This brings in mind DNA transcription and replication as a process proceeding along a DNA strand parallel to the monopole flux tube as a sequence of SFRs involving direct contact between DNA strand and enzymes catalyzing the process and also of corresponding flux tubes. An interesting possibility is that these quantum cobordisms appear routinely in biochemistry of the fundamental linear bio-molecules such as DNA, RNA, tRNA, and amino-acids [K3, K7, K22, K1, K24, K9] [L12].

The quantum cobordism of 2-knot is possible only in ZEO, where the quantum state as a time= constant snapshot is replaced with a superposition of space-time surfaces.

7.5 Could the existence of exotic smooth structures pose problems for TGD?

The article of Gabor Etesi [A2] (<https://cutt.ly/2Md7JWP>) gives a good idea about the physical significance of the existence of exotic smooth structures and how they destroy the cosmic censorship hypothesis (CCH of GRT stating that spacetimes of GRT are globally hyperbolic so that there are no time-like loops).

7.5.1 Smooth anomaly

No compact smoothable topological 4-manifold is known, which would allow only a single smooth structure. Even worse, the number of exotics is infinite in every known case! In the case of non-compact smoothable manifolds, which are physically of special interest, there is no obstruction against smoothness and they typically carry an uncountable family of exotic smooth structures.

One can argue that this is a catastrophe for classical general relativity since smoothness is an essential prerequisite for tensor analysis and partial differential equations. This also destroys hopes that the path integral formulation of quantum gravitation, involving path integral over all possible space-time geometries, could make sense. The term anomaly is certainly well-deserved.

Note however that for 3-geometries appearing as basic objects in Wheeler's superspace approach, the situation is different since for $D < 3$ there is only a single smooth structure. If one has holography, meaning that 3-geometry dictates 4-geometry, it might be possible to avoid the catastrophe.

The failure of the CCH is the basic message of Etesi's article. Any exotic R^4 fails to be globally hyperbolic and Etesi shows that it is possible to construct exact vacuum solutions representing curved space-times which violate the CCH. In other words, GRT is plagued by causal anomalies.

Etesi constructs a vacuum solution of Einstein's equations with a vanishing cosmological constant which is non-flat and could be interpreted as a pure gravitational radiation. This also represents one particular aspect of the energy problem of GRT: solutions with gravitational radiation should not be vacua.

1. Etesi takes any exotic R^4 which has the topology of $S^3 \times R$ and has an exotic smooth structure, which is not a Cartesian product. Etesi maps R^4 to CP_2 , which is obtained from C^2 by gluing CP_1 to it as a maximal ball B_r^3 for which the radial Eguchi-Hanson coordinate approaches infinity: $r \rightarrow \infty$. The exotic smooth structure is induced by this map. The image

of the exotic atlas defines atlas. The metric is that of CP_2 but $SU(3)$ does not act as smooth isometries anymore.

2. After this Etesi performs Wick rotation to Minkowskian signature and obtains a vacuum solution of Einstein's equations for any exotic smooth structure of R^4 .

In TGD, the question of exotic smoothness is encountered both at the level of embedding space and associated fixed spaces and at the level of space-time surfaces and their 6-D twistor space analogies. Could TGD solve the smooth anomaly?

7.5.2 Can embedding space and related spaces have exotic smooth structure?

One can first worry about the exotic smooth structures possibly associated with the M^4 , CP_2 , $H = M^4 \times CP_2$, causal diamond $CD = cd \times CP_2$, where cd is the intersection of the future and past directed light-cones of M^4 , and with M^8 . One can also worry about the twistor spaces CP_3 resp. $SU(3)/U(1) \times U(1)$ associated with M^4 resp. CP_2 .

The key assumption of TGD is that all these structures have maximal isometry groups so that they relate very closely to Lie groups, whose unique smooth structures are expected to determine their smooth structures.

1. The first sigh of relief is that all Lie groups have the standard smooth structure. In particular, exotic R^4 does not allow translations and Lorentz transformations as isometries. I dare to conclude that also the symmetric spaces like CP_2 and hyperbolic spaces such as $H^n = SO(1, n)/SO(n)$ are non-exotic since they provide a representation of a Lie group as isometries and the smoothness of the Lie group is inherited. This would mean that the charts for the coset space G/H would be obtained from the charts for G by an identification of the points of charts related by action of subgroup H .

Note that the mass shell H^3 , as any 3-surface, has a unique smooth structure by its dimension.

2. Second sigh of relief is that twistor spaces CP_3 and $SU(3)/U(1) \times U(1)$ have by their isometries and their coset space structure a standard smooth structure.

In accordance with the vision that the dynamics of fields is geometrized to that of surfaces, the space-time surface is replaced by the analog of twistor space represented by a 6-surface with a structure of S^2 bundle with space-time surface X^4 as a base-space in the 12-D product of twistor spaces of M^4 and CP_2 and by its dimension $D = 6$ can have only the standard smooth structure unless it somehow decomposes to $(S^3 \times R) \times R^2$. Holography of smoothness would prevent this since it has boundaries because X^4 as base space has boundaries at the boundaries of CD .

If exotic smoothness is allowed at the space-time level in the proposed sense ordinary smooth structure could be possible at the level of twistor space in the complement of a Cartesian product of the fiber space S^2 with a discrete set of points associated with partonic 2-surfaces.

3. cd is an intersection of future and past directed light-cones of M^4 . Future/past directed light-cone could be seen as a subset of M^4 and implies standard smooth structure is possible. Coordinate atlas of M^4 is restricted to cd and one can use Minkowski coordinates also inside the cd . cd could be also seen as a pile of light-cone boundaries $S^2 \times R_+$ and by its dimension $S^2 \times R$ allows only one smooth structure.
4. M^8 is a subspace of complexified octonions and has the structure of 8-D translation group, which implies standard smooth structure.

The conclusion is that continuous symmetries of the geometry dictate standard smoothness at the level of embedding space and related structures.

7.5.3 Could TGD eliminate the smoothness anomaly or provide a physical interpretation for it?

The question of exotic smoothness is encountered both at the level of embedding space and associated fixed spaces and at the level of space-time surfaces and their 6-D twistor space analogies.

What does the induction of a differentiable structure really mean? Here my naive expectations turn out to be wrong. If a sub-manifold $S \subset H$ can be regarded as an embedding of smooth manifold N to $S \subset H$, the embedding $N \rightarrow S \subset H$ induces a smooth structure in S (<https://cutt.ly/tMtvG79>). The problem is that the smooth structure would not be induced from H but from N and for a given 4-D manifold embedded to H one could also have exotic smooth structures. This induction of smooth structure is of course physically adhoc.

It is not possible to induce the smooth structure from H to sub-manifold. The atlas defining the smooth structure in H cannot define the charts for a sub-manifold (surface). For standard R^4 one has only one atlas.

1. Could holography of smoothness make sense in the general case?

The first trial to get rid of exotics was based on the holography of smoothness and did not involve TGD. Could a smooth structure at the boundary of a 4-manifold could dictate that of the manifold uniquely. Could one speak of holography for smoothness? Manifolds with boundaries would have the standard smooth structure.

1. The obvious objection is that the coordinate atlas for 3-D boundary cannot determine 4-D atlas in any way because the boundary cannot have information of the topology of the interior.
2. The holography for smoothness is also argued to fail (<https://cutt.ly/3MewY0t>). Assume a 4-manifold W with 2 different smooth structures. Remove a ball B^4 belonging to an open set U and construct a smooth structure at its boundary S^3 . Assume that this smooth structure can be continued to W . If the continuation is unique, the restrictions of the 2 smooth structures in the complement of B^4 would be equivalent but it is argued that they are not.
3. The first layman objection is that the two smooth structures of W are equivalent in the complement $W - B^3$ of an arbitrary small ball $B^3 \subset W$ but not in the entire W . This would be analogous to coordinate singularity. For instance, a single coordinate chart is enough for a sphere in the complement of an arbitrarily small disk.

An exotic smooth structure would be like a local defect in condensed matter physics. In fact it turned out that this intuitive idea is correct: it can be shown that the exotic smooth structures are equivalent with standard smooth structure in a complement of a set having co-dimension zero (<https://cutt.ly/7MbGqx2>). This does not save the holography of smoothness in the general case but gives valuable hints for how exotic smoothness might be realized in TGD framework.

2. Could holography of smoothness make sense in the TGD framework?

Could $M^8 - H$ duality and holography make holography of smoothness possible in the TGD framework?

1. In the TGD framework space-time is 4-surface rather than abstract 4-manifold. 4-D general coordinate invariance, assuming that 3-surfaces as generalization of point-like particles are the basic objects, suggests a fully deterministic holography. A small failure of determinism is however possible and expected, and means that space-time surfaces analogous to Bohr orbits become fundamental objects. Could one avoid the smooth anomaly in this framework?

The 8-D embedding space topology induces 4-D topology. My first naive intuition was that the 4-D smooth structure, which I believed to be somehow inducible from that of $H = M^4 \times CP_2$, cannot be exotic so that in TGD physics the exotics could not be realized. But can one really exclude the possibility that the induced smooth structure could be exotic as a 4-D smooth structure?

2. In the TGD framework and at the level of $H = M^4 \times cP_2$, one can argue that the holography implied by the general coordinate invariance somehow determines the smooth structure in the interior of space-time surface from the coordinate atlas at the boundary. One would have a holography of smoothness. It is however not obvious why this unique structure should be the standard one.
3. One has also holography in M^8 and this induces holography in H by $M^8 - H$ duality. The 3-surfaces X^3 inducing the holography in M^8 are parts of mass shells, which are hyperbolic spaces $H^3 \subset M^4 \subset M^8$. 3-surfaces X^3 could be even hyperbolic 3-manifolds as unit cells of tessellations of H^3 . These hyperbolic manifolds have unique smooth structures as manifolds with dimension $D < 4$.

The hypothesis is that one can assign to these 3-surfaces a 4-surface by a number theoretic dynamics requiring that the normal space is associative, that is quaternionic [L14, L15]. The additional condition is that the normal space contains commutative subspace makes it possible to parametrize normal spaces by points of CP_2 . $M^8 - H$ duality would map a given normal space to a point of CP_2 . $M^8 - H$ duality makes sense also for the twistor lift.

4. A more general statement would be as follows. A set of 3-surfaces as sub-manifolds of mass shells H_m^3 determined by the roots of polynomial P having interpretation as mass square values defining the 4-surface in M^8 take the role of the boundaries. Mass-shells H_m^3 or partonic 2-surfaces associated with them having particle interpretation could correspond to discontinuities of derivatives and even correspond to failure of manifold property analogous to that occurring for Feynman diagrams so that the holography of smoothness would decompose to a piece-wise holography.

The regions of $X^4 \subset M^8$ connecting two sub-sequent mass shells would have a unique smooth structure induced by the hyperbolic manifolds H^3 at the ends.

It is important to notice that the holography of smoothness does not force the smooth 4-D structure to be the standard one.

3. Could the exotic smooth structures have a physical interpretation in the TGD framework?

In the TGD framework, exotic smooth structures could also have a physical interpretation. As noticed, the failure of the standard smooth structure can be thought to occur at a point set of dimension zero and correspond to a set of point defects in condensed matter physics. This could have a deep physical meaning.

1. The space-time surfaces in $H = M^4 \times CP_2$ are images of 4-D surfaces of M^8 by $M^8 - H$ -duality. The proposal is that they reduce to minimal surfaces analogous to soap films spanned by frames. Regions of both Minkowskian and Euclidean signature are predicted and the latter correspond to wormhole contacts represented by CP_2 type extremals. The boundary between the Minkowskian and Euclidean region is a light-like 3-surface representing the orbit of partonic 2-surface identified as wormhole throat carrying fermionic lines as boundaries of string world sheets connecting orbits of partonic 2-surfaces.
2. These fermionic lines are counterparts of the lines of ordinary Feynman graphs, and have ends at the partonic 2-surfaces located at the light-like boundaries of CD and in the interior of the space-time surface. The partonic surfaces, actually a pair of them as opposite throats of wormhole contact, in the interior define topological vertices, at which light-like partonic orbits meet along their ends.
3. These points should be somehow special. Number theoretically they should correspond points with coordinates in an extension of rationals for a polynomial P defining 4-surface in H and space-time surface in H by $M^8 - H$ duality. What comes first in mind is that the throats touch each other at these points so that the distance between Minkowskian space-time sheets vanishes. This is analogous to singularities of Fermi surface encountered in topological condensed matter physics: the energy bands touch each other. In TGD, the partonic 2-surfaces at the mass shells of M^4 defined by the roots of P are indeed analogs

of Fermi surfaces at the level of $M^4 \subset M^8$, having interpretation as analog of momentum space.

Could these points correspond to the defects of the standard smooth structure in X^4 ? Note that the branching at the partonic 2-surface defining a topological vertex implies the local failure of the manifold property. Note that the vertices of an ordinary Feynman diagram imply that it is not a smooth 1-manifold.

4. Could the interpretation be that the 4-manifold obtained by removing the partonic 2-surface has exotic smooth structure with the defect of ordinary smooth structure assignable to the partonic 2-surface at its end. The situation would be rather similar to that for the representation of exotic R^4 as a surface in CP_2 with the sphere at infinity removed [A2].
5. The failure of the cosmic censorship would make possible a pair creation. As explained, the fermionic lines can indeed turn backwards in time by going through the wormhole throat and turn backwards in time. The above picture suggests that this turning occurs only at the singularities at which the partonic throats touch each other. The QFT analog would be as a local vertex for pair creation.
6. If all fermions at a given boundary of CD have the same sign of energy, fermions which have returned back to the boundary of CD, should correspond to antifermions without a change in the sign of energy. This would make pair creation without fermionic 4-vertices possible.

If only the total energy has a fixed sign at a given boundary of CD, the returned fermion could have a negative energy and correspond to an annihilation operator. This view is nearer to the QFT picture and the idea that physical states are Galois confined states of virtual fundamental fermions with momentum components, which are algebraic integers. One can also ask whether the reversal of the arrow of time for the fermionic lines could give rise to gravitational quantum computation as proposed in [A8].

4. A more detailed model for the exotic smooth structure associated with a topological 3-vertex

One can ask what happens to the 4-surface near the topological 3-particle vertex and what is the geometric interpretation of the point defect. The first is whether the description of the situation is possible both in M^8 and H . Here one must consider momentum conservation.

1. By Uncertainty Principle and momentum conservation at the level of M^8 , the incoming real momenta of the particle reaction are integers in the scale defined by CD. In the standard QFT picture, the momenta at the vertex of physical particles are at different mass shells.

In M^8 picture, the mass squared values of virtual fermions are in general algebraic and also complex roots of a polynomial defining the 3-D mass shells H_m^3 of $M^4 \subset M^8$, determining 4-surface by associative holography.

In the standard wave mechanical picture assumed also in TGD, a given topological vertex, describable in terms of partonic 2-surfaces, would correspond to a multi-local vertex in M^8 in accordance with the representation of a local n-vertex in M^4 as convolution of n-local vertices in momentum space realizing momentum conservation.

2. M^8-H duality maps M^4 momenta by inversion to positions in $M^4 \subset H$. This encourages the question whether the topological vertex could be described also in M^8 as a partonic surface at single algebraic mass shell in M^8 , mapped by M^8-H duality to a single $a = \text{constant}$ hyperboloid in $M^4 \subset H$.

The virtual momenta at the level of M^8 are algebraic, in general complex, integers. The algebraic mass squared values at the mass shell of M^8 would be the same for all particles of the vertex. This kind of correspondence does not make sense if M^8-H duality applies to the full algebraic momenta. The assumption has been that it applies to the rational parts of the momenta.

3. The rational parts of the algebraic integer valued 4-momenta of virtual fermions are in general not at the same mass shell. Could this make possible a description in terms of partonic 2-surfaces at fixed mass *resp.* $a = \text{constant}$ shell at the level of M^8 *resp.* H ?

The classical space-time surface in H , partonic 2-surfaces and fermion lines at them are characterized by classical momenta by Noether's theorem. Quantum classical correspondence, realized in ZEO as Bohr orbitology, suggests that the classical 4-momenta assignable to these objects correspond to the rational parts of the momenta at M^8 mass shell. Could the rational projections of M^8 momenta at H_n^3 correspond to different mass squared values at given H^3 ?

4. Note that this additional symmetry for complexified momentum space and position space descriptions would be analogous to the duality of twistor amplitudes position space and the space of area momenta.

How to describe the topological vertex in H ? The goal is to understand how exotic smooth structure and its point defects could emerge from this picture. The physical picture applied hitherto is as follows.

1. 3 partonic orbits meet at a vertex described by a partonic 2-surface. Assume that they are located to single $a = \text{constant } H^3 \subset M^4 \subset H$.
2. The partonic wormhole throats appear as pairs at the opposite Minkowskian space-time sheets. There are three pairs corresponding to 3 external particle lines and one line which must be a bosonic line describing fermion-antifermion bound state disappears: this corresponds to a boson absorption (or emission).

The opposite throats carry opposite magnetic monopole charges. The only possibility, not noticed before, is that the opposite wormhole throats for the partonic orbit, which ends at the vertex, must coincide at the vertex. The minimal option is that the exotic smooth structure is associated with this partonic orbit turning back in time. The two partonic orbits, which bind 4-D Euclidean regions as wormhole throats, would fuse to a larger 4-D surface with an exotic smooth structure.

Fermion-antifermion annihilation occurs at a point at which fermion and antifermion lines meet. The first guess is that this point corresponds to the defect of the smooth structure.

3. There is an analogy with the construction of Etesi [A2] in which a homologically non-trivial ball CP_1 glued to the C^2 at infinity to construct an exotic smooth structure. One dimension disappears for the glued 3-surface at infinity.

In the partonic vertex, one has actually two homologically non-trivial 2-surfaces with opposite homology charges as boundaries between wormhole contact and Minkowskian regions and they fuse together in the partonic vertex. Also now, one dimension disappears as the partonic 2-surfaces become identical so that 3-D wormhole contact contracts to single 2-D partonic 2-surface.

4. The defect for the smooth structure associated with the fusion of the pair of wormhole orbits should correspond to a point at which fermion and antifermion lines meet.

This suggests that the throats do not fuse instantaneously but gradually. The fusion would start from a single touching point identifiable as the fermion-antifermion vertex, serving as a seed of a phase transition, and would proceed to the entire wormhole contact so that it reduces to a partonic 2-surface.

One can argue that one has a problem if this surface is homologically non-trivial. Could the process make the closed partonic 2-surface homologically trivial. A simplified example is the fusion of two circles with opposite winding numbers ± 1 on a cylinder. The outcome is two homologically non-trivial circles of opposite orientations on top of each other. The phase transition starting from a point would correspond to a touching of the circles.

A couple of further comments are in order.

1. The connection of the pair of wormhole throats to the associative holography is an interesting question. The 4-D tangent planes of $X^4 \subset M^8$ mass shell correspond to points of CP_2 . They would be different at the two parallel sheets.

At the mass shell H_m^3 the branches would coincide. The presence of two tangent planes could give rise to two different holographic orbits, which coincide at the initial mass shell and gradually diverge from each other just as in the above model for the fusion of partonic 2-surfaces. The failure of the strict determinism for the associative holography at the partonic 2-surface would make in TGD the analogy of fermion-antifermion annihilation vertex possible.

2. There is also an analogy with the cusp catastrophe in which the projection of the cusp catastrophe as a 2-surface in 3-D space with behavior variable x and two control parameters (a, b) has a boundary at which two real roots of a polynomial of degree 3 coincide. The projection to the (a, b) plane gives a sharp shape, whose boundary is a V-shaped curve in which the sides of V become parallel at the vertex. The vertex corresponds to maximal criticality. The particle vertex would be a critical phenomenon in accordance with the interpretation as a phase transition.

8 A possible connection with family replication phenomenon?

In TGD framework the genus g of the partonic 2-surfaces is proposed to label fermion families [K2, K4, K5]. One can characterize by genus g the topology of light-like partonic orbits and identify the three fermion generators as 2-surfaces with genus $g = 0, 1, 2$ with the special property that they are always hyper-elliptic. Quantum mechanically also topological mixing giving rise to CKM mixing is possible. The view is that given connected 3-surface can contain several light-like 3-surface with different genera. For instance, hadrons would be such surfaces.

There are however questions to be answered.

1. The genera $g = 0, 1, 2$ assigned with the free fermion families correspond to Riemann surfaces, which are always hyper-elliptic allowing therefore Z_2 as a global conformal symmetry. These complex curves correspond to degrees $n = 2, 3, 4$ for the corresponding polynomials. For $n \leq 4$ can write explicit solutions for the roots of the polynomials. Could there be a deep connection between particle physics and mathematical cognition?
2. The homology and genus for 2-surfaces of CP_2 correlate with each other [A5]: is this consistent with the proposed topologization of color hypercharge implying color confinement?
3. $h_{eff}/h = n$ hypothesis means that dark variant of particle particle characterized by genus g is n -fold covering of this surface. In the general case the genus of covering is different. Is this consistent with the genus-generation correspondence?
4. The degree of complex curve correlates with the genus of the curve. Is generation-genus correspondence consistent with the assumption that partonic 2-surfaces have algebraic curve as CP_2 projection (this need not be the case)?

8.1 How the homology charge and genus correlate?

Complex surfaces in CP_2 are highly interesting from TGD point of view.

1. The model for elementary particles assumes that the partonic 2-surfaces carrying fermion number are homologically non-trivial, in other words they carry Kähler magnetic monopole flux having values $q = \pm 1$ and $q = \pm 2$. The idea is that color hyper charge $Y = \{\pm 2/3, \pm 1/3\}$ is proportional to n for quarks and color confinement topologizes to the vanishing of total homology charge [K5].
2. The explanation of the family replication phenomenon [K2] in terms of genus-generation correspondence states that the three quarks and lepton generations correspond to the three lowest genera $g = 0, 1, 2$ for partonic 2-surfaces. Only these genera are always hyper-elliptic allowing thus a global Z_2 conformal symmetry. The physical vision is that for higher genera the handles behave like free particles. Is this proposal consistent with the proposal for the topologization of color confinement?

There is a result [A5] (page 124) stating that if the homology charge q is divisible by 2 then one must have $g \geq q^2/4 - 1$. If q is divisible by h , which is odd power of prime, one has $g \geq (q^2/4 - 1) - (q^2/4h^2)$. For $q = 2$ the theorem allows $g \geq 0$ so that all genera with color hypercharge $Y = \pm 2/3$ are realized.

The theorem says however nothing about $q = 0, 1$. These charges can be assigned to the two different geodesic spheres of CP_2 with $g = 0$ remaining invariant under $SO(3)$ and $U(2)$ subgroups of $SU(3)$ respectively. Is $g > 0$ possible for $q = 1$ as the universality of topological color confinement would require? For $q = 3$ one would have $g \geq 1$. For $q = 4$ $h = 2$ divides q and one has $g \geq 2$. It would seem $g \geq 5$. The conditions become more restrictive for higher q , which suggests that for $q = 0, 1$ one has $g \geq 0$ so that the topologization of color hypercharge would make sense.

8.2 Euler characteristic and genus for the covering of partonic 2-surface

Hierarchy of Planck constants $h_{eff}/h = n$ means a hierarchy of space-time surfaces identifiable as n -fold coverings. The proposal is that the number of sheets in absence of singularities is maximal possible and equals to the dimension of the extension dividing the order of its Galois group.

The Euler characteristic of n -fold covering in absence of singular points is $\chi_n = n\chi$. If there are singular (ramified) points these give a correction term given by Riemann-Hurwitz formula (see <http://tinyurl.com/y7n2acub>.)

In absence of singularities one has from $\chi = -2(g - 1)$ and $\chi_n = n\chi$

$$g_n = n(g - 1) + 1 . \quad (8.1)$$

For $n = 1$ this indeed gives $g_1 = g$ independent of g . One can also combine this with the formula $g = (d - 1)(d - 2)/2$ holding for non-singular algebraic curves of degree d .

Singularities are unavoidable at algebraic points of cognitive representations at which some subgroup of Galois group leaves the point invariant (say rational point in ordinary sense). One can consider the possibility that fermions are located at the singular points at which several sheets of covering touch each other. This would give a correction factor to the formula. If the projection map from the covering to based is of form $\Pi(z) = z^n$ at the singular point P , one says that singularity has ramification index $e_P = n$ and the algebraic genus would increase to

$$g_n = n(g - 1) + 1 + \frac{1}{2} \sum_P (e_P - 1) . \quad (8.2)$$

Indeed, singularities mean that sheets touch each other at singular points and this increases connectivity.

Under what conditions the genus of dark partonic surface with $n > 1$ can be same as that of the ordinary partonic surface representing visible matter? For the genera $g = 0$ and $g = 1$ this is possible so that these genera would be in an exceptional role also from the point of view of dark matter.

1. For $g = 1$ one has $g_n = g = 1$ independent of n in absence of singular point. Torus topology (assignable to muon and (c,s) quarks) is exceptional. In presence of singularities the genus would increase by the $\sum_P (e_P - 1)/2$ independent of the value of n . The lattice of points for elliptic surfaces would suggest existence of infinite number of singular points if the abelian group operations preserve the singular character of the points so that the genus would become infinite.
2. For $g = 0$ one would have $g_n = -n + 1$ in absence of singularities. Only $n = 1$ - ordinary matter - is possible without singularities. Dark matter is however possible if singularities are allowed. For sphere one would obtain $g_n = -n + 1 + \sum_P (e_P - 1)/2 \geq 0$. The condition $n \leq \sum_P (e_P - 1)/2 + 1$ must therefore hold true for $g \geq 0$.

The condition $g_n = -n + 1 + \sum_P (e_P - 1)/2 = g = 0$ gives $\sum_P (e_P - 1) = 2(n - 1)$. For spherical topology it is possible to have dense set of rational points so that it is possible create cognitive representations with arbitrary number of points which can be also singular. One might argue that this kind of situation corresponds to a non-perturbative phase.

- For $g = 2$ one would have $g_n = n + 1 + \sum_P (e_P - 1)/2$ and genus would grow with n even in absence of singularities and would be very large for large values of h_{eff} . $g_n = 2$ is obtained with $n = 1$ (ordinary matter) and no singular points not even allowed for $n = 1$. $g_n = g = 2$ is not possible for $n > 1$.

Note that dark $g \geq 2$ fermions cannot correspond to lower generation fermions with singular points of covering. More generally, one could say that $g \geq 2$ fermions can exist only with standard value of Planck constant unless they are singular coverings of $g < 2$ fermions.

What is clear that the model of dark matter predicts breaking of universality. This breaking is not seen in the standard model couplings but makes it visible in a more delicate manner and might allow to understand why the masses of fermions increase with generation index.

8.3 All genera are not representable as non-singular algebraic curves

Suppose for a moment that partonic 2-surfaces correspond to rational maps of algebraic curves in CP_2 to M^4 that is deformations of these curves in M^4 direction. This assumption is of course questionable but deserves to be studied.

The formula (for algebraic curve see <http://tinyurl.com/nt6tkey>)

$$g = \frac{(d-1)(d-2)}{2} + \frac{\sum \delta_s}{2} ,$$

where $\delta_s > 0$ characterizes the singularity, does not allow all genera for algebraic curves for $\sum \delta_s = 0$: one has $g = 0, 0, 1, 3, 6, 10, \dots$ for $d = 1, 2, \dots$

For instance, $g = 2$, which would correspond in TGD to third quark or lepton generation is not possible without singularities for $d = 3$ curve having $g = 1$ without singularities!

This raises questions. Could the third fermion generation actually correspond to $g = 3$? Or does it correspond to $g = 2$ 2-surface of CP_2 , which is more general surface than algebraic curve meaning that it is not representable as complex surface? Or could third generation fermions correspond to $g = 0$ or $g = 1$ curves with singular point of covering by Galois group so that several sheets touch each other?

To sum up, if the results for algebraic varieties generalize to TGD framework, they suggest notable differences between different fermion families. Universality of standard model interactions says that the only differences between fermion families are due to the different masses. It is not clear whether the different masses could be due to the differences at number theoretical level and dark matter sectors.

- All genera can appear as ordinary matter ($d = 1$). Dark variants of $g = 1$ states have $g_d = 1$ automatically in absence of singular points. Dark variants of $g = 0$ states must have singular point in order to give $g_n = 0$. Dark variants of $g = 2$ states with $g_d = 2$ are obtained from $g = 1$ states with singularities. The special role of the two lowest is analogous to their special role for algebraic curves.
- If one assumes that partonic 2-surfaces correspond to algebraic curves, one obtains again that $g = 2$ surfaces must correspond to singular $g = 0$ and $g = 1$ which could be dark in TGD sense.

9 Summary and future prospects

In the following I give a brief summary about what has been done. I concentrate on $M^8 - H$ duality since the most significant results are achieved here.

It is fair to say that the new view answers the following a long list of open questions.

- When $M^8 - H$ correspondence is true (to be honest, this question emerged during this work!)? What are the explicit formulas expressing associativity of the tangent space or normal space of the 4-surface?

The key element is the formulation in terms of complexified $M^8 - M_c^8$ - identified in terms of octonions and restriction $M_c^8 \rightarrow M^8$. One loses the number field property but for polynomials ring property is enough. The level surfaces for real and imaginary parts of octonionic polynomials with real coefficients define 4-D surfaces in the generic case.

Associativity condition is an additional condition reducing the dimension of the space-time surface unless some components of $RE(P)$ or $IM(P)$ are critical meaning that also their gradients vanish. This conforms with the quantum criticality of TGD and provides a concrete first principle realization for it.

An important property of $IM(P_1P_2)$ is its linearity with respect to $IM(P_i)$ implying that this condition gives the surfaces $IM(P_i) = 0$ as solutions. This generalizes by induction to $IM(P_1P_2\dots P_n)$. For $RE(P_1P_2) = 0$ linearity does not hold true and there is a genuine interaction. A physically attractive idea is that $RE(P_1P_2) = 0$ holds true inside CDs and for wormhole contacts between space-time sheets with Minkoskian signature. One can generalize this also to $IM(P_1/P_2)$ and $RE(P_1/P_2)$ if rational functions are allowed. Note however that the origins of octonionic coordinates in P_i must be on the octonionic real line.

2. How this picture corresponds to twistor lift? The twistor lift of Kähler action (dimensionally reduced Kähler action in twistor space of space-time surface) one obtains two kinds of space-time regions. The regions, which are minimal surfaces and obey dynamics having no dependence on coupling constants, correspond naturally to the critical regions in M^8 and H .

There are also regions in which one does not have extremal property for both Kähler action and volume term and the dynamics depends on coupling constant at the level of H . These regions are associative only at their 3-D ends at boundaries of CD and at partonic orbits, and the associativity conditions at these 3-surfaces force the initial values to satisfy the conditions guaranteeing preferred extremal property. The non-associative space-time regions are assigned with the interiors of CDs. . The particle orbit like space-time surfaces entering to CD are critical and correspond to external particles.

It has later turned out [L11] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

3. The surprise was that $M^4 \subset M^8$ is naturally co-associative. If associativity holds true also at the level of H , $M^4 \subset H$ must be associative. This is possible if $M^8 - H$ duality maps tangent space in M^8 to normal space in H and vice versa.
4. The connection to the realization of the preferred extremal property in terms of gauge conditions of subalgebra of SSA is highly suggestive. Octonionic polynomials critical at the boundaries of space-time surfaces would determine by $M^8 - H$ correspondence the solution to the gauge conditions and thus initial values and by holography the space-time surfaces in H .
5. A beautiful connection between algebraic geometry and particle physics emerges. Free many-particle states as disjoint critical 4-surfaces can be described by products of corresponding polynomials satisfying criticality conditions. These particles enter into CD, and the non-associative and non-critical portions of the space-time surface inside CD describe the interactions. One can define the notion of interaction polynomial as a term added to the product of polynomials. It can vanish at the boundary of CD and forces the 4-surface to be connected inside CD. It also spoils associativity: interactions are switched on. For bound states the coefficients of interaction polynomial are such that one obtains a bound state as associative space-time surface.
6. This picture generalizes to the level of quaternions. One can speak about 2-surfaces of space-time surface with commutative or co-commutative tangent space. Also these 2-surfaces would be critical. In the generic case commutativity/co-commutativity allows only 1-D curves.

At partonic orbits defining boundaries between Minkowskian and Euclidian space-time regions inside CD the string world sheets degenerate to the 1-D orbits of point like particles at their boundaries. This conforms with the twistorial description of scattering amplitudes in terms of point like fermions.

For critical space-time surfaces representing incoming states string world sheets are possible as commutative/co-commutative surfaces (as also partonic 2-surfaces) and serve as correlates for (long range) entanglement) assignable also to macroscopically quantum coherent system ($h_{eff}/h = n$ hierarchy implied by adelic physics).

7. The octonionic polynomials with real coefficients form a commutative and associative algebra allowing besides algebraic operations function composition. Space-time surfaces therefore form an algebra and WCW has algebra structure. This could be true for the entire hierarchy of Cayley-Dickson algebras, and one would have a highly non-trivial generalization of the conformal invariance and Cauchy-Riemann conditions to their n-linear counterparts at the n :th level of hierarchy with $n = 1, 2, 3, ..$ for complex numbers, quaternions, octonions,... One can even wonder whether TGD generalizes to this entire hierarchy!
8. In the original version of this article I did not realize that there are two options for realizing the idea that the M_c^4 projection of space-time surface in M_c^8 must belong to M^4 .
 - (a) I proposed that the *projection* from M_c^8 to real M^4 (for which M^1 coordinate is real and E^3 coordinates are imaginary with respect to i !) defines the real space-time surface mappable by $M^8 - H$ duality to CP_2 [L6].
 - (b) An alternative option, which I have not considered in the original versions of [L6, L8] is that only the roots of the 4 vanishing polynomials as coordinates of M_c^4 belong to M^4 so that m^0 would be real root and m^k , $k = 1, ..., 3$ imaginary with respect to $i \rightarrow -i$. M_c^8 coordinates would be invariant (“real”) under combined conjugation $i \rightarrow -i, I_k \rightarrow -I_k$. In the following I will speak about this property as *Minkowskian reality*. This could make sense. Outside CD these conditions would not hold true. This option looks more attractive than the first one. Why these condition can be true just inside CD, should be understood.
9. The use of polynomials or rational functions could be also an approximation. Analytic functions of real variable extended to octonionic functions would define the most general space-time surfaces but the limitations of cognition would force to use polynomial approximation. The degree n of the polynomial determining also $h_{eff} = nh_0$ would determine the quality of the approximation and at the same time the “IQ” of the system.

All big pieces of quantum TGD are now tightly interlinked.

1. The notion of causal diamond (CD) and therefore also ZEO can be now regarded as a consequence of the number theoretic vision and $M^8 - H$ correspondence, which is also understood physically.
2. The hierarchy of algebraic extensions of rationals defining evolutionary hierarchy corresponds to the hierarchy of octonionic polynomials.
3. Associative varieties for which the dynamics is critical are mapped to minimal surfaces with universal dynamics without any dependence on coupling constants as predicted by twistor lift of TGD. The 3-D associative boundaries of non-associative 4-varieties are mapped to initial values of space-time surfaces inside CDs for which there is coupling between Kähler action and volume term.
4. Free many particle states as algebraic 4-varieties correspond to product polynomials in the complement of CD and are associative. Inside CD the addition of interaction terms vanishing at its boundaries spoils associativity and makes these varieties connected.

5. The super variant of the octonionic algebraic geometry makes sense, and one obtains a beautiful correlation between the fermion content of the state and corresponding space-time variety. This suggests that twistorial construction indeed generalizes. Criticality for the external particles giving rise to additional constraints on the coefficients of polynomials could make possible to have well-define summation over corresponding varieties.

What mathematical challenges one must meet?

1. One should prove more rigorously that criticality is possible without the reduction of dimension of the space-time surface.
2. One must demonstrate that SSA conditions can be true for the images of the associative regions (with 3-D or 4-D). This would obviously pose strong conditions on the values of coupling constants at the level of H .

Concerning the description of interactions there are several challenges.

1. Do associative space-time regions have minimal surface extremals as images in H and indeed obeying universal critical dynamics? As found, the study of the known extremals supports this view.
2. Could one construct the scattering amplitudes at the level of M^8 ? Here the possible problems are caused by the exponents of action (Kähler action and volume term) at H side. Twistorial construction [K18] however leads to a proposal that the exponents actually cancel. This happens if the scattering amplitude can be thought as an analog of Gaussian path integral around single extremum of action and conforms with the integrability of the theory. In fact, nothing prevents from defining zero energy states in this manner! If this holds true then it might be possible to construct scattering amplitudes at the level of M^8 .
3. What about coupling constants? Coupling constants make themselves visible at H side both via the vanishing conditions for Noether charges in sub-algebra of SSA and via the values of the non-vanishing Noether charges. $M^8 - H$ correspondence determining the 3-D boundaries of interaction regions within CDs suggests that these couplings must emerge from the level M^8 via the criticality conditions posing conditions on the coefficients of the octonionic polynomials coding for interactions.

Could all coupling constant emerge from the criticality conditions at the level of M^8 ? The ratio of R^2/l_P^2 of CP_2 scale and Planck length appears at H level. Also this parameter should emerge from $M^8 - H$ correspondence and thus from criticality at M^8 level. Physics would reduce to a generalization of the catastrophe theory of Rene Thom!

4. The description of interactions at the space-time surface associated with single CD should be M^8 counterpart of the H picture in which 3 light-like partonic orbits meet at common end topological vertex - defined by a partonic 2-surface and fermions scatter without touching. Now one has octonionic sparticle lines and interaction vertex becomes possible. This conforms with the idea that interactions take place at discrete points belonging to the extension of rationals. The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$, which satisfy $RE(P) = IM(P) = 0$ and are singular and doubly critical. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

5. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be at real line (time axis) in the octonionic sense, and guarantees the associativity and commutativity of the polynomials. Arbitrary CDs cannot

be located along this line. Can one assume that all CDs involved with *observable* processes satisfy this condition?

If not, how do the 4-varieties associated with octonionic polynomials with different origins interact? How could one avoid losing the extremely beautiful associative and commutative algebra? It seems that one cannot form their products and sums and must form the Cartesian product of M^8 :s with different tips for CDS and formulate the interaction in this framework. In the case of space-time surfaces associated with different CDs the discrete intersections of space-time surfaces would define the interaction vertices.

6. Super-octonionic geometry suggests that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in an appropriate extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of super-twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic: indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N} = 4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the extension of rationals considered. The rest would be dictated by symmetries and integrations over various moduli spaces, which should be number theoretically universal so that residue calculus strongly suggests itself.

7. What is the connection with super conformal variant of Yangian symmetry, whose generalization in TGD framework is highly suggestive? Twistorial construction of scattering amplitudes at the level of M^8 looks highly promising idea and could also realize Yangian supersymmetry. The conjecture is that the twistorial amplitudes decompose to M^4 and CP_2 parts with similar structure with E^4 spin (electroweak isospin) replacing ordinary spin and that the integrands in Grassmannians emerging from the conservation of M^4 and E^4 4-momenta are identical in the two cases and thus guarantee Yangian supersymmetry in both sectors. The only difference would be due to the product of delta functions associated with the “negative helicities” (weak isospins with negative sign) expressible as a delta function in the complement of $SU(3)$ Cartan algebra $U(1) \times U(1)$ by using exponential map.

It is appropriate to close with a question about fundamentals.

1. The basic structure at M^8 side consists of complexified octonions. The metric tensor for the complexified inner product for complexified octonions (no complex conjugation with respect to i for the vectors in the inner product) can be taken to have any signature $(\epsilon_1, \dots, \epsilon_8)$, $\epsilon_i = \pm 1$. By allowing some coordinates to be real and some coordinates imaginary one obtains effectively any signature from say purely Euclidian signature. What matters is that the restriction of complexified metric to the allowed sub-space is real. These sub-spaces are linear Lagrangian manifolds for Kähler form representing the commuting imaginary unit i . There is analogy with wave mechanics. Why M^8 -actually M^4 - should be so special real section? Why not some other signature?
2. The first observation is that the CP_2 point labelling tangent space is independent of the signature so that the problem reduces to the question why M^4 rather than some other signature $(\epsilon_1, \dots, \epsilon_4)$. The intersection of real subspaces with different signatures and same origin $(t, r) = 0$ is the common sub-space with the same signature. For instance, for $(1, -1, -1, -1)$ and $(-1, -1, -1, -1)$ this subspace is 3-D $t = 0$ plane sharing with CD the lower tips of CD. For $(-1, 1, 1, 1)$ and $(1, 1, 1, 1)$ the situation is same. For $(1, -1, -1, -1)$ and $(1, 1, -1, -1)$ $z = 0$ holds in the intersection having as common with the lower boundary of CD the boundary of 3-D light-cone. One obtains in a similar manner boundaries of 2-D and 1-D light-cones as intersections.

3. What about CDs in various signatures? For a fully Euclidian signature the counterparts for the interiors of CDs reduce to 4-D intervals $t \in [0, T]$ and their exteriors and thus the space-time varieties representing incoming particles reduce to pairs of points $(t, r) = (0, 0)$ and $(t, r) = (T, 0)$: it does not make sense to speak about external particles. For other signatures the external particles correspond to 4-D surfaces and dynamics makes sense. The CDs associated with the real sectors intersect at boundaries of lower dimensional CDs: these lower-dimensional boundaries are analogous to subspaces of Big Bang (BB) and Big Crunch (BC).
4. I have not found any good argument for selecting $M^4 = M^{1,3}$ as a unique signature. Should one allow also other real sections? Could the quantum numbers be transferred between sectors of different signature at BB and BC? The counterpart of Lorentz group acting as a symmetry group depends on signature and would change in the transfer. Conservation laws should be satisfied in this kind of process if it is possible. For instance, in the leakage from $M^4 = M^{1,3}$ to $M^{i,j}$, say $M^{2,2}$, the intersection would be $M^{1,2}$. Momentum components for which signature changes, should vanish if this is true. Angular momentum quantization axis normal to the plane is defined by two axis with the same signature. If the signatures of these axes are preserved, angular momentum projection in this direction should be conserved. The amplitude for the transfer would involve integral over either boundary component of the lower-dimensional CD.

Could the leakage between signatures be detected as disappearance of matter for CDs in elementary particle scales or lab scales?

5. One can also raise a question about the role of WCW geometry as a continuous infinite-D geometry: could the discretization by cognitive representations making WCW effectively discrete mean its loss? It seems that this cannot be the case. At least in the real sector continuum must be present and the discretization reflects only the discreteness of cognitive representations. In principle continuous WCW could make sense also in p-adic sectors of the adèle.

The identification of space-time surfaces as zero loci of polynomials generalizes to rational functions and even transcendental functions although the existence of the p-adic counterparts of these functions requires additional conditions. Could one interpret the representation in terms of polynomials and possibly rational functions as an approximation? Could the hierarchy of approximations obtained in this manner give rise to a hierarchy of hyper-finite factors of type II_1 defining a hierarchy of measurement resolutions [K23]?

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